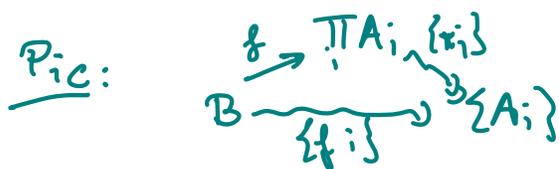


Recall:

Def 2.2.35 Let \mathcal{C} be a loc. small cat and consider a family of objects $\{A_i\}_{i \in I}$ in \mathcal{C} . The product of the A_i is an object in \mathcal{C} denoted by $\prod_{i \in I} A_i$, together with arrows $\pi_j \in \text{Hom}_{\mathcal{C}}(\prod_{i \in I} A_i, A_j) \forall j \in I$, the projections satisfying the following universal property: given any object B in \mathcal{C} and arrows $f_i: B \rightarrow A_i \forall i$, there exists a unique arrow f such that

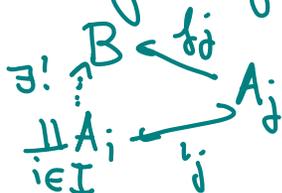


When I is finite, then write $A_1 \times \dots \times A_n$ for $\prod_{i=1}^n A_i$.

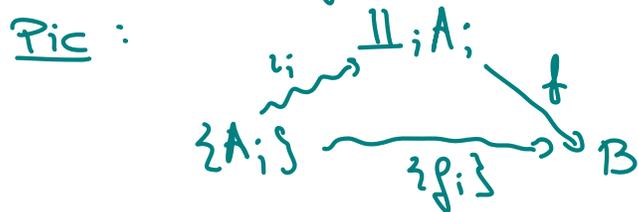


The dual notion:

Def 2.2.36 (coproduct) Let \mathcal{C} be a locally small cat. and consider a family of objects $\{A_i\}_{i \in I}$ in \mathcal{C} . The coproduct of the A_i is an object in \mathcal{C} , denoted $\coprod_{i \in I} A_i$, together with arrows $\iota_j \in \text{Hom}_{\mathcal{C}}(A_j, \coprod_{i \in I} A_i)$ for each j , satisfying the following universal property: given any object $B \in \mathcal{C}$ and arrows $f_i: A_i \rightarrow B \forall i \in I$, the following diag. commutes

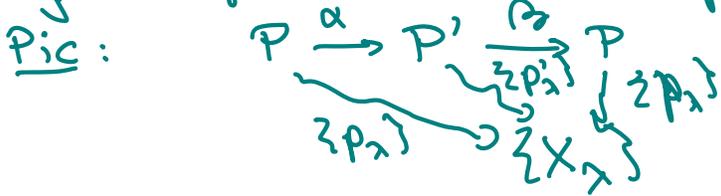


When I is finite, then $A_1 \coprod \dots \coprod A_n$ is written.



Thm 2.2.37 If $(P, \{p_\lambda: P \rightarrow X_\lambda\}_{\lambda \in \Lambda})$ and $(P', \{p'_\lambda: P' \rightarrow X_\lambda\}_{\lambda \in \Lambda})$ are both products for the same family of objects $\{X_\lambda\}_{\lambda \in \Lambda}$ in a cat \mathcal{C} , then there exists an isom. $\alpha: P \xrightarrow{\sim} P'$ s.t. $p'_\lambda \circ \alpha = p_\lambda \forall \lambda \in \Lambda$. The analogous statement holds for coproducts.

Pf: We prove the statement for products.



Since $(P, \{p_\lambda\})$ is a product and $(P', \{p'_\lambda\})$ is an object with maps to each X_λ , there is a unique map $\beta: P' \rightarrow P$ s.t. $p_\lambda \circ \beta = p'_\lambda$. Switching roles of P and P' , obtain a unique map $\alpha: P \rightarrow P'$ s.t. $p'_\lambda = p'_\lambda \circ \alpha$. Consider the composition $\beta \circ \alpha: P \rightarrow P$. We have $p_\lambda \circ \beta \circ \alpha = p'_\lambda \circ \alpha = p_\lambda \forall \lambda \in \Lambda$. The identity map $1_P: P \rightarrow P$ also satisfies the condition $p_\lambda \circ 1_P = p_\lambda \forall \lambda$. So by uniqueness of product, $\beta \circ \alpha = 1_P$. Again switching roles, we also get $\alpha \circ \beta = 1_{P'}$. $\Rightarrow \alpha$ is an iso and uniqueness is part of the universal property. \square

\rightarrow Mapping into a product is completely determined by mapping into each of the factors. (dual for coproduct)

The Cartesian product serves as a product in many cats \Rightarrow Grp, Ring, R-Mod, Top, the usual product $\prod_{\lambda \in \Lambda} X_\lambda$ is a product in the resp. cat. (operations coordinatewise)

Ex 2.2.38 Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of sets. The coproduct of $\{X_\lambda\}_{\lambda \in \Lambda}$ in Set is given by the disjoint union with the various inclusion maps.

Thm 2.2.39 Let R be a ring and $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of left

R-modules. A coproduct for the family is given by the direct sum of modules (3)

$$\bigoplus_{\lambda \in \Lambda} M_\lambda = \{ (x_\lambda)_{\lambda \in \Lambda} \mid x_\lambda \neq 0 \text{ for at most finitely many } \lambda \} \\ \subseteq \prod_{\lambda \in \Lambda} M_\lambda.$$

together with the inclusion maps $M_\lambda \xrightarrow{i_\lambda} \bigoplus_{\lambda \in \Lambda} M_\lambda$, sending each m_λ to the tuple $(0, \dots, 0, \underset{\lambda}{\uparrow} m_\lambda, 0, \dots, 0)$.

Pf: Given R-module homs $g_\lambda: M_\lambda \rightarrow N \forall \lambda$, we need to show that \exists a unique R-module hom. $\alpha: \bigoplus_{\lambda \in \Lambda} M_\lambda \rightarrow N$ s.t. $\alpha \circ i_\lambda = g_\lambda$. We define:

$$\alpha((m_\lambda)_{\lambda \in \Lambda}) := \sum_{\lambda \in \Lambda} g_\lambda(m_\lambda)$$

Note that since $(m_\lambda)_{\lambda \in \Lambda}$ is in the direct sum, almost all $m_\lambda = 0$ so sum on RHS is finite, and hence makes sense in N .

We need to check that α is linear:

$$\alpha((m_\lambda)_{\lambda \in \Lambda} + (n_\lambda)_{\lambda \in \Lambda}) = \alpha(m_\lambda + n_\lambda) = \sum g_\lambda(m_\lambda + n_\lambda) \\ = \sum g_\lambda(m_\lambda) + \sum g_\lambda(n_\lambda) = \alpha(m_\lambda) + \alpha(n_\lambda).$$

similar for multiplication.

For uniqueness of α , note that $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is generated by $i_\lambda(m_\lambda)$ for $m_\lambda \in M_\lambda$. Thus, if α' also satisfies $\alpha' \circ i_\lambda = g_\lambda \forall \lambda$, then $\alpha(i_\lambda(m_\lambda)) = g_\lambda(m_\lambda) = \alpha'(i_\lambda(m_\lambda))$, so the maps must be equal. \square

Remark: If $|\Lambda| < \infty$, then $\bigoplus M_\lambda$ and $\prod M_\lambda$ are identical, but product and coprod. are not the same, since one involves projections and the other one inclusions.

Ex 2.2.40: (1) In Top disj. unions serve as coproducts.

(2) In Grp and Grp, coproducts exist and are given as free prods.

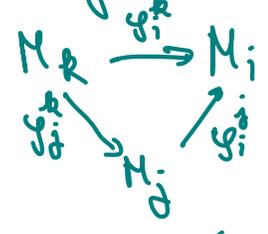
(3) In Ring more complicated \rightsquigarrow can construct coproduct in cat. of comm. rings.

Limits + colimits

Def 2.2.41 Let (I, \geq) be a partially ordered set and let \mathcal{C} be a cat. An inverse system in \mathcal{C} indexed by I is a contravariant functor $\underline{PO}(I) \rightarrow \mathcal{C}$.

Prob: This means: for each $i \in I$ we get an object $M_i \in \mathcal{C}$. Further, in $\underline{PO}(I)$ there is exactly one arrow $i \rightarrow j$ for each $i \leq j$ and the image under the contravariant functor $\underline{PO}(I) \rightarrow \mathcal{C}$ is an arrow $M_j \rightarrow M_i$. Finally, our functor must preserve composition of arrows, so when $k \geq j \geq i$, the arrow $M_k \rightarrow M_i$ should match the composition of arrows through j . Thus an inverse system in \mathcal{C} consists of the following data:

- for each $i \in I$ an object $M_i \in \mathcal{C}$ and
- for each $i \leq j$ an arrow $y_i^j: M_j \rightarrow M_i$ in \mathcal{C} , s.t. whenever $i \leq j \leq k$, the following diagram must commute:



Note moreover that $y_i^i = \text{id}_{M_i}$ (since functors preserve identities)
We write $\{M_i, y_i^j\}$ for an inverse system in \mathcal{C} .

Ex 2.2.42 (a) An inverse system in a cat \mathcal{C} indexed by \mathbb{N} is determined by a diagram of the form

$$X_0 \xleftarrow{\alpha_0} X_1 \xleftarrow{\alpha_1} X_2 \xleftarrow{\alpha_2} \dots$$

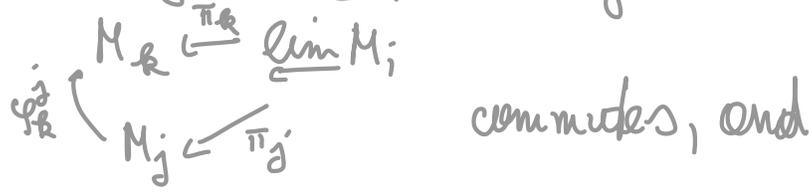
all arrows $X_j \rightarrow X_i$, $j > i$ are given by composition.

(*) Let I be a family of submods of a (left) R -module M . Then I is part. ordered set with reverse inclusion $\geq: L \leq N \Leftrightarrow L \supseteq N$.

If $N \subseteq L$, we have an inclusion map $N \rightarrow L$ and the family of submods I with inclusions forms an inverse system of R -mods
Special case: chain $M_1 \supseteq M_2 \supseteq M_3 \dots$ is inverse system

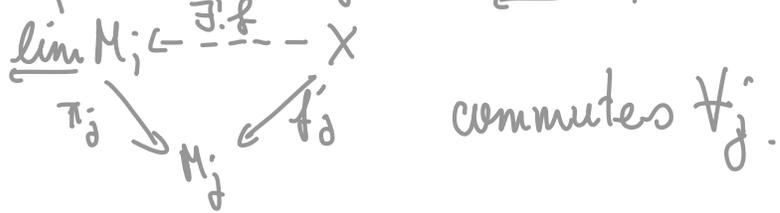
Def 2.2.43 Let \mathcal{C} be a cat and let $\{M_i, \varphi_i^j\}$ be an inverse system on \mathcal{C} indexed by \mathbb{I} . The limit or inverse limit of $\{M_i, \varphi_i^j\}$ consists of an object $\varprojlim M_i$ and arrows $\pi_i: \varprojlim M_i \rightarrow M_i$

called projections s.t. $\forall j \geq k$ in \mathbb{I} , the diagram



satisfies the following universal property: for all arrows $f_i: X \rightarrow M_i$ s.t. $\varphi_i^j \circ f_j = f_i \quad \forall i, j$ (i.e. $\begin{array}{ccc} M_i & \xleftarrow{f_i} & X \\ \varphi_i^j \uparrow & & \swarrow f_j \\ M_j & & \end{array}$ commutes)

there exists a unique arrow $f: X \rightarrow \varprojlim M_i$ s.t.



One can show that $\varprojlim M_i$ exists up to isom \leadsto therefore we can call it the limit.

Prop 1.65: limit is the same as nat. trf $\text{PO}(\mathbb{I}) \begin{array}{c} \xrightarrow{\Delta_L \text{ - const. functor.}} \\ \downarrow \\ \varprojlim \end{array} \mathcal{C}$

Ex 2.2.44 A terminal object can be viewed as a limit of the empty diagram (!)

Thm 2.2.45 Let R be any ring. Every inverse system of left R -modules over any partially ordered set has a limit.

Pf: Let \mathbb{I} be a partially ordered set and consider an inverse system of R -modules indexed by \mathbb{I} , say $\{M_i, \varphi_i^j\}$
 Set $L := \{ (m_i) \in \prod_i M_i : \varphi_i^j(m_j) = m_i \quad \forall i \leq j \}$.

One can show (!) that $L \subseteq \prod_{i \in I} M_i$ is a submodule.

Now for each i , let $\pi_i: L \rightarrow M_i$ be the restriction of the projection map $\prod M_i \rightarrow M_i$ to L .

Claim: L is a limit for the inverse system with proj. maps π_j .

Pf of Claim: Note that $\varphi_i^j \pi_j((m_k)_k) = \varphi_i^j(m_j) = m_i = \pi_i((m_k)_k)$
out of L

So $\varphi_i^j \pi_j = \pi_i$.

Moreover, suppose that we are given an R -module X and R -module homom., $f_i: X \rightarrow M_i$ s.t. $\varphi_i^j f_j = f_i \quad \forall i \leq j$. Define

$$X \xrightarrow{g} \prod M_i \text{ by } x \mapsto (f_i(x))_i.$$

First, note that $\pi_i(g(x)) = \pi_i((f_k(x))_k) = f_i(x) \quad \forall i$. Also, g is an R -module homom., induced by the universal property of the product.

Moreover $\text{im}(g) \subseteq L$, so we can restrict g to an R -module hom.: $f: X \rightarrow L$

($\therefore \forall x \in X: \varphi_i^j(\underbrace{\pi_j(g(x))}_{m_j}) = \varphi_i^j(f_j(x)) = f_i(x) = \underbrace{\pi_i(g(x))}_{m_i}$). This means $g(x) \in L$).

Consider the restriction $f: X \rightarrow L$ given as $f(x) = (f_i(x))_i$.

Finally we check that L and f satisfy the universal property:

• $\varprojlim M_i \xleftarrow{f} X$ commutes: by construction, since $\pi_i(f(x)) = f_i(x) \quad \forall x \in X$.



• f unique: Suppose that h is another R -module hom $X \rightarrow L$ s.t.



also commutes. Given any $x \in X$, let

$$h(x) = (m_i)_i.$$

Then $m_i = \pi_i(h(x)) = f_i(x) \quad \forall i$

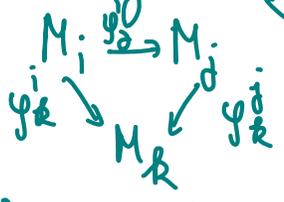
$$\Rightarrow h(x) = (m_i)_i = (f_i(x))_i = f(x) \Rightarrow h = f. \quad \square$$

Remark One can adapt the proof to show that limits in Set exist.

The dual construction is the colimit:

Def 2.2.46 Let (I, \geq) be a partially ordered set and \mathcal{C} be a cat. A direct system in \mathcal{C} indexed by I is a covariant functor $PO(I) \rightarrow \mathcal{C}$.

Prob Again this def can be unneeded: a direct system consists of the following data: • for each $i \in I$, an object M_i in \mathcal{C} and • for each $i \leq j$ an arrow $\varphi_j^i: M_i \rightarrow M_j$ in \mathcal{C} s.t. $\forall i \leq j \leq k$ the following diag. commutes



Note: $\varphi_i^i = id_{M_i}$. We say $\{M_i, \varphi_j^i\}$ is a direct system.

Ex 2.2.47 (a) a direct system in \mathcal{C} indexed by \mathbb{N} is det. by a diag. $X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \xrightarrow{\alpha_3} \dots$

all other arrows $X_i \rightarrow X_j$, $i < j$ are given by composition.

(b) \mathcal{I} family of submods of an R -module M , ordered by inclusion. Whenever $N \leq L$, we have an inclusion map $N \rightarrow L$ and $\{N, \subseteq\}$ forms a direct system.

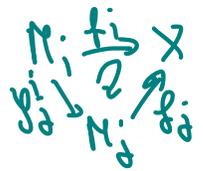
Def 2.2.48 Let \mathcal{C} be a cat and let $\{M_i, \varphi_j^i\}$ be a direct system on \mathcal{C} indexed by I . The colimit or direct limit of $\{M_i, \varphi_j^i\}$ consists of an object $\varinjlim M_i$

and arrows $\alpha_i: M_i \rightarrow \varinjlim M_i$

called injection arrows s.t. $\alpha_j \varphi_j^i = \alpha_i \quad \forall i, j \in I$

satisfying the universal property: \forall arrow $f_i: M_i \rightarrow X$ s.t. $f_j \varphi_j^i = f_i$

$\forall i, j \in I$



, there exists a unique arrow

$f: \varinjlim M_i \rightarrow X$ s.t. $\varinjlim M_i \xrightarrow{\exists! f} X$ commutes.

Again one can show that $\varinjlim M_i$, if it exists, is unique up to isom. In fact, it is the initial object in some appropriate (and technical) category. So we can refer to the colimit of a direct system.

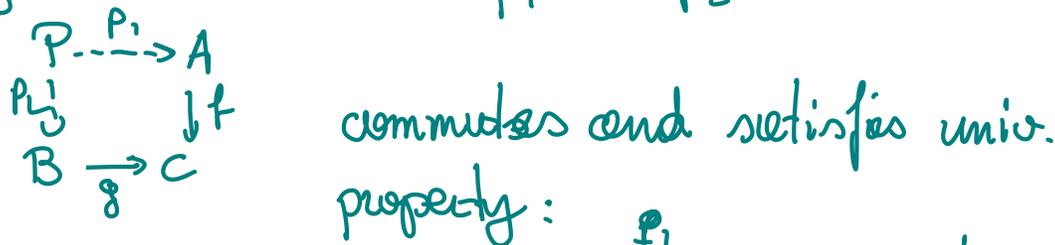
Rmk: Can describe limit as nat. trf. to constant functor.

Thm 2.2.49 Let R be any ring. Every direct system of left R -mods over any part. ordered set has a limit.

Pf: See [yife Thm 1.76].

Many constructions arise as (co-) limits:

Def 2.2.50 Let \mathcal{C} be a cat. A pullback of the arrows f and g consists of an object P and arrows p_1 and p_2 s.t.



\forall objects Q and arrows q_1, q_2 s.t. $\begin{array}{ccc} Q & \xrightarrow{q_1} & A \\ q_2 \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$ commutes,

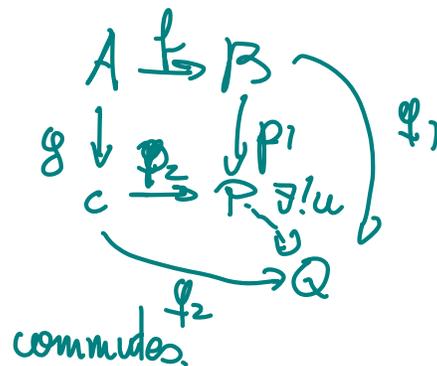
there exists a unique u s.t. $\begin{array}{ccc} Q & \xrightarrow{u} & P \xrightarrow{p_1} A \\ q_2 \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$ commutes

We say that $\begin{array}{ccc} P & \xrightarrow{p_1} & A \\ p_2 \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$ is a pullback diagram limit

Dually: Let \mathcal{C} be a cat. A pushout of the arrows f, g

consists of object P and arrows p_1 and p_2 s.t.

(9)



We say that $\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow p_1 \\ C & \xrightarrow{f_2} & P \end{array}$ is a pushout diagram.
colimit

Adjoint functors

Def 2.2.51 Let \mathcal{C} and \mathcal{D} be locally small categories. Two covariant functors $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ form an adjoint pair (F, G) if given any objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, there is a bijection between the Hom sets

$$\text{Hom}_{\mathcal{D}}(F(C), D) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(C, G(D))$$

which is natural on both objects, i.e., $\forall f \in \text{Hom}_{\mathcal{C}}(C_1, C_2)$ and $g \in \text{Hom}_{\mathcal{D}}(D_1, D_2)$ the diagrams

$$\text{Hom}_{\mathcal{D}}(F(C_1), D) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(C_1, G(D))$$

$$\downarrow F(f)^*$$

$$\text{Hom}_{\mathcal{D}}(F(C_2), D) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(C_2, G(D))$$

and

$$\text{Hom}_{\mathcal{D}}(F(C), D_1) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(C, G(D_1))$$

$$\downarrow G(g)_*$$

$$\text{Hom}_{\mathcal{D}}(F(C), D_2) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(C, G(D_2))$$

commute $\forall C \in \mathcal{C}, D \in \mathcal{D}$. We say that F is left adjoint (10) to G , or F has a right adjoint, and that G is the right adjoint of F , or that G has a left adjoint.

Examples 2.2.52

(1) ← Most important ex. in hom. alg! (Hom-Tensor adjointness)

Let R, S be rings and M an R - S -bimodule (i.e. M left R -mod. and right S -module s.t. $\forall r \in R, s \in S, m \in M: (r \cdot m) \cdot s = r \cdot (m \cdot s)$)

Then $\text{Hom}_R(M \otimes_S X, Y) \cong \text{Hom}_S(X, \text{Hom}_R(M, Y))$

for X S -module, Y R -module. So $(M \otimes_S -, \text{Hom}_R(M, -))$

$F: S\text{-Mod} \rightarrow R\text{-Mod}$
 $X \mapsto M \otimes_S X$
 $G: R\text{-Mod} \rightarrow S\text{-Mod}$
 $Y \mapsto \text{Hom}_R(M, Y)$

$\mathcal{C} = S\text{-mod}$
 $\mathcal{D} = R\text{-mod}$

is an adjoint pair between S -mods and R -mods

$\text{Hom}_R(M, Y)$ has left S -mod structure: (1)
 $(s \cdot f)(m) = f(ms)$ (1)

(2) Fix a ring R . Consider the functor

$\text{Free}: \underline{\text{Set}} \rightarrow R\text{-Mod}$
 $I \mapsto R^I = \bigoplus_I R$

$\underline{\text{Set}} \xrightarrow{\text{Free}} R\text{-Mod}$
 Forget

sending a set I to the free module indexed by I .

This functor is a left adjoint to the forgetful functor $R\text{-Mod} \rightarrow \underline{\text{Set}}$ i.e., there is a natural bijection

$\text{Hom}_{R\text{-Mod}}(\text{Free}(I), M) \cong \text{Hom}_{\underline{\text{Set}}}(I, M)$

identify image of forget with M

(in general, one can define a free module on a set I by the univ. property: given a function f from a set I to an R -module M , $\exists!$ R -module hom $\bigoplus_I R \rightarrow M$ that agrees with f on basis elements)

Exercise: left and right adjoints are unique up to nat. isom.

§ 2.3. The category of chain complexes

(11)

Finally we can introduce the cat of chain complexes, and talk about exact seq. and homology (and the snake lemma, of course!)

Maps of Chain complexes

↪ category, but need right notion for maps between c.s. ↪ want homology to be functorial.

Def 2.3.1 Let $(F_\bullet, \partial_\bullet^F)$ and $(G_\bullet, \partial_\bullet^G)$ be two complexes. A map of complexes or a chain map, written $h: (F_\bullet, \partial_\bullet^F) \rightarrow (G_\bullet, \partial_\bullet^G)$ or simply $h: F \rightarrow G$, is a sequence of homomorphisms between R -modules $h_n: F_n \rightarrow G_n$ s.t. the following diagram commutes:

$$\begin{array}{ccccccc} \dots & \longrightarrow & F_{n+1} & \longrightarrow & F_n & \longrightarrow & F_{n-1} & \longrightarrow & \dots \\ & & \downarrow h_{n+1} & & \downarrow h_n & & \downarrow h_{n-1} & & \\ \dots & \longrightarrow & G_{n+1} & \longrightarrow & G_n & \longrightarrow & G_{n-1} & \longrightarrow & \dots \end{array}$$

This means: $h_n \partial_{n+1}^F = \partial_{n+1}^G h_{n+1} \quad \forall n.$

Note here: we say "map" although we mean R -module hom. here!

Ex 2.3.2 The 0 and id-maps $(F_\bullet, \partial_\bullet) \rightarrow (F_\bullet, \partial_\bullet)$ are:

$$O_F: F_n \rightarrow F_n \quad \text{and} \quad \text{id}_F: F_n \rightarrow F_n \quad \forall n.$$

$x \mapsto 0$ $x \mapsto x$

Def 2.3.3 Let R be a ring. The category of chain complexes of R -modules, denoted $\text{Ch}(R\text{-Mod})$ or simply $\text{Ch}(R)$ has objects: all c.s. of R -modules and arrows: all maps of c.s. of R -modules.

When $R = \mathbb{Z}$, write $\text{Ch}(\underline{\text{Ab}})$ for $\text{Ch}(\mathbb{Z})$, the cat of chain c.s. of abelian groups.

Note that id-map above are precisely the identity arrows

in $\text{Ch}(R)$.

(12)

Exercise Show that the isomorphisms in the cat $\text{Ch}(R)$ are precisely the maps of α .

$$\begin{array}{ccccc} \rightarrow F_{n+1} & \xrightarrow{\partial_{n+1}^F} & F_n & \rightarrow & F_{n-1} \\ h_{n+1} \downarrow & g_{n+1} \downarrow & h_n \downarrow & & h_{n-1} \downarrow \\ G_{n+1} & \xrightarrow{\partial_{n+1}^G} & G_n & \rightarrow & G_{n-1} \end{array} \quad \text{s.t. } h_n \text{ is an iso } \forall n.$$

\Leftarrow : If all h_n are isos: $\exists! u_n: G_n \rightarrow F_n$ s.t. $u_n \circ h_n = 1_{F_n}$
 $h_n \circ u_n = 1_{G_n}$

The $\{u_n\}$ are chain maps: $u_n \circ \partial_{n+1}^G = \partial_{n+1}^F \circ u_{n+1}$

Know: $h_n \circ \partial_{n+1}^F = \partial_{n+1}^G \circ h_{n+1}$: $u_n \circ \underbrace{\partial_{n+1}^G \circ h_{n+1}}_{1_{G_{n+1}}} \circ u_{n+1} = \underbrace{u_n \circ h_n}_{1_{F_n}} \circ \partial_{n+1}^F \circ u_{n+1} = \partial_{n+1}^F \circ u_{n+1}$ ✓

Clearly $\{u_n\}$ and $\{h_n\}$ are inverse to each other \Rightarrow iso in $\text{Ch}(R)$.

\Rightarrow : If $\{u_n\}, \{h_n\}$ are inverse chain maps then $h_n \circ u_n = 1_{G_n}$ by def.
 $u_n \circ h_n = 1_{F_n}$

by def all u_n, h_n are R -module homs \Rightarrow isom. of R -mods.

This is a good notion of maps of α : it induces homomorphisms in homology, which allows us to say that homology is a functor.

Lemma 2.34 Let $h: (F_\bullet, \partial_\bullet^F) \rightarrow (G_\bullet, \partial_\bullet^G)$ be a map of α . For all n , h_n restricts to homom. $B_n(h): B_n(F_\bullet) \rightarrow B_n(G_\bullet)$ and $Z_n(h): Z_n(F_\bullet) \rightarrow Z_n(G_\bullet)$
 \Rightarrow It induces homom. on homology $H_n(h): H_n(F_\bullet) \rightarrow H_n(G_\bullet)$.

\Downarrow : Since $h_n \circ \partial_{n+1}^F = \partial_{n+1}^G \circ h_{n+1}$, any element $a \in B_n(F_\bullet)$, say $a = \partial_{n+1}^F(b)$ is taken to $h_n(a) = h_n \circ \partial_{n+1}^F(b) = \partial_{n+1}^G(h_{n+1}(b)) \in \text{im}(\partial_{n+1}^G) = B_n(G_\bullet)$.

Similarly, if $a \in Z_n(F_\bullet) = \text{ker } \partial_n^F$, then

$$\mathcal{D}_n^{\mathbb{Q}} h_n(\alpha) = h_{n-1} \mathcal{D}_n^{\mathbb{F}}(\alpha) = h_{n-1}(0) = 0.$$

(13)

$\Rightarrow h_n(\alpha) \in \ker(\mathcal{D}_n^{\mathbb{Q}}) = Z_n(G_0)$. Finally, the restriction of h_n to $Z_n(\mathbb{F}) \rightarrow Z_n(G_0)$ sends $B_n(\mathbb{F}_0)$ into $B_n(G_0)$, and thus it induces a well-def. homom. of the quotients $H_n(\mathbb{F}_0) \rightarrow H_n(G_0)$.