WEEK 12

Recall:

Def 2.2.35 Let $\mathcal{C}$ be a loc. small cat and consider a family of objects $\{A_i\}_{i\in I}$ in $\mathcal{C}$. The product of the A_i is an object in $\mathcal{C}$ denoted by $\prod_{i\in I} A_i$, togethe with arrows $\pi_j \in \mathrm{Hom}_{\mathcal{C}}(\prod_i A_i, A_j)$ $\forall j \in I$, the projections satisfying the following universal property: given any object B in $\mathcal{C}$ and arrows $f_i : B \to A_i$ $\forall i$, there exists a unique arrow f such that

$$\begin{tikzcd} B \arrow[d, dashed, "\exists! f"'] \arrow[dr, "f_j"] & \\ \prod_j A_j \arrow[r, "\pi_j"] & A_j \end{tikzcd}$$

commutes $\forall j$

When I is finite, then write $A_1 \times \ldots \times A_n$ for $\prod_{i=1}^n A_i$.

Pic:

$$\begin{tikzcd} & \prod_i A_i \arrow[dr, squiggly, "\{\pi_i\}"] & \\ B \arrow[ur, "f"] \arrow[rr, squiggly, "\{f_i\}"'] & & \{A_i\} \end{tikzcd}$$

The dual notion:

Def 2.2.36 (coproduct) Let $\mathcal{C}$ be a locally small cat. and consider a family of objects $\{A_i\}_{i\in I}$ in $\mathcal{C}$. The coproduct of the A_i is an object in $\mathcal{C}$, denoted $\coprod_{i\in I} A_i$, together with arrows $\iota_j \in \mathrm{Hom}_{\mathcal{C}}(A_j, \coprod_i A_i)$ for each j, satisfying the following universal property: given any object $B \in \mathcal{C}$ and arrows $f_i : A_i \to B$ $\forall i \in I$, the following diag. commutes

$$\begin{tikzcd} B & \\ \coprod_{i\in I} A_i \arrow[u, dashed, "\exists!"] & A_j \arrow[ul, "f_j"'] \arrow[l, "\iota_j"] \end{tikzcd}$$

When I is finite, then $A_1 \amalg \ldots \amalg A_n$ is written.

Pic:

$$\begin{tikzcd} & \coprod_i A_i \arrow[dr, "f"] & \\ \{A_i\} \arrow[ur, squiggly, "\iota_i"] \arrow[rr, squiggly, "\{f_i\}"'] & & B \end{tikzcd}$$

Thm 2.2.37 If $(P, \{p_\lambda : P \to X_\lambda\}_{\lambda\in\Lambda})$ and $(P', \{p'_\lambda : P' \to X_\lambda\}_{\lambda\in\Lambda})$ are both products for the same family of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in a cat $\mathcal{C}$, then there exists an isom. $\alpha : P \xrightarrow{\sim} P'$ s.t. $p'_\lambda \circ \alpha = p_\lambda$ $\forall \lambda\in\Lambda$. The analogous statement holds for coproducts.

(2)

Pf: We prove the statement for products.

Pic:

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[drr, bend right, "\{p_\lambda\}"'] & P' \arrow[r, "\beta"] \arrow[dr, "\{p'_\lambda\}"] & P \arrow[d, "\{p_\lambda\}"] \\ & & \{X_\lambda\} \end{tikzcd}$$

Since $(P, \{p_\lambda\})$ is a product and $(P', \{p'_\lambda\})$ is an object with maps to each X_λ, there is a unique map $\beta: P' \to P$ s.t. $p_\lambda \circ \beta = p'_\lambda$. Switching roles of P and P', obtain a unique map $\alpha: P \to P'$ s.t. $p_\lambda = p'_\lambda \circ \alpha$. Consider the composition $\beta\circ\alpha : P \to P$. We have $p_\lambda \circ \beta \circ \alpha = p'_\lambda \circ \alpha = p_\lambda$ $\forall \lambda \in \Lambda$. The identity map $1_P : P \to P$ also satisfies the condition $p_\lambda \circ 1_P = p_\lambda$ $\forall \lambda$. So by uniqueness of product, $\beta\circ\alpha = 1_P$. Again switching roles, we also get $\alpha\circ\beta = 1_{P'}$. $\Rightarrow \alpha$ is an iso and uniqueness is part of the universal property. $\square$

→ Mapping <u>into</u> a product is completely determined by mapping into each of the factors. (dual for coproduct)

The Cartesian product serves as a product in many cats

⇝ <u>Grp</u>, <u>Ring</u>, <u>R-Mod</u>, <u>Top</u>, the usual product $\prod_{\lambda\in\Lambda} X_\lambda$ is a product in the resp. cat. (operations coordinatewise)

Ex 2.2.38 Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of sets. The coproduct of $\{X_\lambda\}_{\lambda\in\Lambda}$ in <u>Set</u> is given by the disjoint union with the various inclusion maps.

Thm 2.2.39 Let R be a ring and $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of left

R-modules. A coproduct for the family is given by the direct sum of modules

$$\bigoplus_{\lambda\in\Lambda} M_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} \mid x_\lambda \neq 0 \text{ for at most finitely many } \lambda\} \subseteq \prod_{\lambda\in\Lambda} M_\lambda .$$

together with the inclusion maps $M_\lambda \xrightarrow{i_\lambda} \bigoplus_{\lambda\in\Lambda} M_\lambda$, sending each m_λ to the tuple $(0,\dots,0,\underset{\uparrow_\lambda}{m_\lambda},0,\dots,0)$.

Pf: Given R-module homs $g_\lambda : M_\lambda \to N$ $\forall\lambda$, we need to show that $\exists$ a unique R-module hom. $\alpha : \bigoplus_{\lambda\in\Lambda} M_\lambda \to N$ s.t. $\alpha\circ i_\alpha = g_\lambda$. We define:

$$\alpha((m_\lambda)_{\lambda\in\Lambda}) := \sum_{\lambda\in\Lambda} g_\lambda(m_\lambda)$$

Note that since $(m_\lambda)_{\lambda\in\Lambda}$ is in the direct sum, almost all $m_\lambda = 0$ so sum on RHS is finite, and hence makes sense in N.
We need to check that α is linear:

$$\alpha((m_\lambda)_{\lambda\in\Lambda} + (n_\lambda)_{\lambda\in\Lambda}) = \alpha(m_\lambda + n_\lambda) = \sum g_\lambda(m_\lambda + n_\lambda)$$
$$= \sum g_\lambda(m_\lambda) + \sum g_\lambda(n_\lambda) = \alpha(m_\lambda) + \alpha(n_\lambda).$$

similar for multiplication.
For uniqueness of α, note that $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is generated by $i_\lambda(m_\lambda)$ for $m_\lambda \in M_\lambda$. Thus, if α' also satisfies $\alpha'\circ i_\lambda = g_\lambda$ $\forall\lambda$, then $\alpha(i_\lambda(m_\lambda)) = g_\lambda(m_\lambda) = \alpha'(i_\lambda(m_\lambda))$, so the maps must be equal. $\square$

Rmk: If $|\Lambda| < \infty$, then $\bigoplus M_\lambda$ and $\prod M_\lambda$ are identical, but product and coprod. are not the same, since one involves projections and the other one inclusions.

Ex 2.2.40: (1) In Top disj. unions serve as coproducts.
(2) In Sgrp and Grp, coproducts exist and are given as free prods.
(3) In Ring more complicated ⇝ can construct coproduct in cat. of comm. rings.

<u>Def 2.2.41</u> Let $(I, \geq)$ be a partially ordered set and let $\mathcal{C}$ be a cat. An <u>inverse system</u> in $\mathcal{C}$ indexed by I is a contravariant functor $\underline{PO}(\underline{I}) \to \mathcal{C}$.

<u>Rmk</u>: This means: for each $i \in I$ we get an object $M_i \in \mathcal{C}$. Further, in $\underline{PO}(I)$ there is exactly one arrow $i \to j$ for each $i \leq j$ and the image under the contravariant functor $\underline{PO}(I) \to \mathcal{C}$ is an arrow $M_j \to M_i$. Finally, our functor must preserve composition of arrows, so whenever $k \geq j \geq i$, the arrow $M_k \to M_i$ should match the composition of arrows through j. Thus an inverse system in $\mathcal{C}$ consists of the following data:
- for each $i \in I$ an object $M_i \in \mathcal{C}$ and
- for each $i \leq j$ an arrow $\varphi_i^j : M_j \to M_i$ in $\mathcal{C}$, s.t.

whenever $i \leq j \leq k$, the following diagram must commute:

$$\begin{tikzcd} M_k \arrow[rr, "\varphi_i^k"] \arrow[dr, "\varphi_j^k"'] & & M_i \\ & M_j \arrow[ur, "\varphi_i^j"'] & \end{tikzcd}$$

Note moreover that $\varphi_i^i = \mathrm{id}_{M_i}$ (since functors preserve identities)
We write $\{M_i, \varphi_i^j\}$ for an inverse system in $\mathcal{C}$.

<u>E.g. 2.2.42</u> (a) An inverse system in a cat $\mathcal{C}$ indexed by $\mathbb{N}$ is determined by a diagram of the form

$$X_0 \xleftarrow{a_0} X_1 \xleftarrow{a_1} X_2 \xleftarrow{a_2} \cdots$$

all arrows $X_j \to X_i$, $j > i$ are given by composition.

(b) Let I be a family of submods of a (left) R-module M. Then I is part. ordered set with reverse inclusion $\supseteq$: $L \leq N \iff L \supseteq N$. If $N \subseteq L$, we have an inclusion map $N \to L$ and the family of submods I with inclusions forms an inverse system of R-mods
Special case: chain $M_1 \supseteq M_2 \supseteq M_3 \ldots$ is inverse system

Def 2.2.43 Let $\mathcal{C}$ be a cat and let $\{M_i, \varphi_i^j\}$ be an inverse system on $\mathcal{C}$ indexed by I. The limit or inverse limit of $\{M_i, \varphi_i^j\}$ consists of an object $\varprojlim M_i$ and arrows $\pi_i : \varprojlim M_i \to M_i$ called projections s.t. $\forall j \geq k$ in I, the diagram

$$\begin{tikzcd} M_k & \varprojlim M_i \arrow[l, "\pi_k"'] \arrow[dl, "\pi_j"] \\ M_j \arrow[u, "\varphi_k^j"] & \end{tikzcd}$$

commutes, and

satisfies the following universal property: for all arrows $f_i : X \to M_i$ s.t. $\varphi_i^j f_j = f_i$ $\forall i,j$ (i.e.

$$\begin{tikzcd} M_i & X \arrow[l, "f_i"'] \arrow[dl, "f_j"] \\ M_j \arrow[u, "\varphi_i^j"] & \end{tikzcd}$$

commutes)

there exists a unique arrow $f : X \to \varprojlim M_i$ s.t.

$$\begin{tikzcd} \varprojlim M_i \arrow[dr, "\pi_j"'] & & X \arrow[ll, dashed, "\exists! f"'] \arrow[dl, "f_j"] \\ & M_j & \end{tikzcd}$$

commutes $\forall j$.

One can show that $\varprojlim M_i$ exists up to isom $\rightsquigarrow$ therefore we can call it *the* limit.

Rmk 1.65: limit is the same as nat. trf

$L := \varprojlim M_i$ $\quad$ $\Delta L \leftarrow$ const. functor

$$\begin{tikzcd} PO(I) \arrow[r, bend left, "\Delta L"] \arrow[r, bend right, "\varphi"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{C} \end{tikzcd}$$

Ex 2.2.44 A terminal object can be viewed as a limit of the empty diagram (:)

Thm 2.2.45 Let R be any ring. Every inverse system of left R-modules over any partially ordered set has a limit.

Pf: Let I be a partially ordered set and consider an inverse system of R-modules indexed by I, say $\{M_i, \varphi_i^j\}$

Set $L := \{ (m_i) \in \prod_i M_i : \varphi_i^j(m_j) = m_i \ \ \forall i \leq j \}$.

One can show (!) that $L \subseteq \prod_{i\in I} M_i$ is a submodule.
Now for each i, let $\pi_i : L \to M_i$ be the restriction of the projection map $\prod_i M_i \to M_i$ to L.

Claim: L is a limit for the inverse system with proj. maps π_j.
Pf of Claim: Note that $\varphi_i^j \pi_j((m_k)_k) = \varphi_i^j(m_j) \underset{\text{def of } L}{=} m_i = \pi_i((m_k)_k)$

So $\varphi_i^j \pi_j = \pi_i$.
Moreover, suppose that we are given an R-module X and R-module homs, $f_i : X \to M_i$ s.t. $\varphi_i^j f_j = f_i$ $\forall i \le j$. Define

$$X \xrightarrow{g} \prod_i M_i \text{ by } x \mapsto (f_i(x))_i .$$

First, note that $\pi_i(g(x)) = \pi_i((f_k(x))_k) = f_i(x)$ $\forall i$. Also, g is an R-module homom, induced by the universal property of the product.
Moreover $\mathrm{im}(g) \subseteq L$, so we can restrict g to an R-module hom: $f : X \to L$
($\because \forall x \in X$: $\varphi_i^j(\underbrace{\pi_j(g(x))}_{m_j}) = \varphi_i^j(f_j(x)) = f_i(x) = \underbrace{\pi_i(g(x))}_{m_i}$. This means $g(x) \in L$).

Consider the restriction $f : X \to L$ given as $f(x) = (f_i(x))_i$.
Finally we check that L and f satisfy the universal property:

- $$\begin{tikzcd} \varprojlim M_i \arrow[dr, "\pi_i"'] & & X \arrow[ll, dashed, "f"'] \arrow[dl, "f_i"] \\ & M_i & \end{tikzcd}$$ commutes: by construction, since $\pi(f(x)) = f_i(x)$ $\forall x \in X$.

- f unique: Suppose that h is another R-module hom $X \to L$ s.t.
$$\begin{tikzcd} \varprojlim M_i \arrow[dr, "\pi_i"'] & & X \arrow[ll, "h"'] \arrow[dl, "f_i"] \\ & M_i & \end{tikzcd}$$
also commutes. Given any $x \in X$, let $h(x) = (m_i)_i$.
Then $m_i = \pi_i(h(x)) = f_i(x)$ $\forall i$
$\Rightarrow h(x) = (m_i)_i = (f_i(x))_i = f(x) \Rightarrow h = f$. □

Rmk One can adapt the proof to show that limits in Set exist.

The dual construction is the colimit:

<u>Def</u> 2.2.46 Let $(I, \geq)$ be a partially ordered set and $\mathcal{C}$ be a cat. A <u>direct system</u> in $\mathcal{C}$ indexed by I is a covariant functor <u>PO</u>$(I) \to \mathcal{C}$.

<u>Rmk</u> Again this def can be unraveled: a direct system consists of the following data: • for each $i \in I$, an object M_i in $\mathcal{C}$ and • for each $i \leq j$ an arrow $\varphi^i_j : M_i \to M_j$ in $\mathcal{C}$ s.t. $\forall\, i \leq j \leq k$ the following diag. commutes

$$\begin{tikzcd} M_i \arrow[r, "\varphi^i_j"] \arrow[dr, "\varphi^i_k"'] & M_j \arrow[d, "\varphi^j_k"] \\ & M_k \end{tikzcd}$$

Note: $\varphi^i_i = \mathrm{id}_{M_i}$. We say $\{M_i, \varphi^i_j\}$ is a direct system.

<u>Ex 2.2.47</u> (a) a direct system in $\mathcal{C}$ indexed by $\mathbb{N}$ is det. by a diag. $X_1 \xrightarrow{a_1} X_2 \xrightarrow{a_2} X_3 \xrightarrow{a_3} \cdots$
all other arrows $X_i \to X_j$, $i<j$ are given by composition.
(b) I family of submods of an R-module M, ordered by inclusion. Whenever $N \subseteq L$, we have an inclusion map $N \to L$ and $\{N, \subseteq\}$ forms a direct system.

<u>Def 2.2.48</u> Let $\mathcal{C}$ be a cat and let $\{M_i, \varphi^i_j)$ be a direct system on $\mathcal{C}$ indexed by I. The <u>colimit</u> or <u>direct limit</u> of $\{M_i, \varphi^i_j\}$ consists of an object $\varinjlim M_i$

and arrows $\alpha_i : M_i \to \varinjlim M_i$

called <u>insertion arrows</u> s.t. $\alpha_j \varphi^i_j = \alpha_i \quad \forall i,j \in I$
satisfying the universal property: $\forall$ arrow $f_i : M_i \to X$ s.t. $f_j \varphi^i_j = f_i$ $\forall i,j,\ i \leq j$

$$\begin{tikzcd} M_i \arrow[r, "f_i"] \arrow[dr, "\varphi^i_j"'] & X \\ & M_j \arrow[u, "f_j"'] \end{tikzcd}$$

, there exists a unique arrow

$f: \varinjlim M_i \to X$ s.t.

$$\begin{tikzcd} \varinjlim M_i \arrow[r, dashed, "\exists! f"] & X \\ & M_j \arrow[ul, "\alpha_j"] \arrow[u, "f_j"'] \end{tikzcd}$$

commutes.

Again one can show that $\varinjlim M_i$, if it exists, is unique up to isom. In fact, it is the initial object in some appropriate (and technical) category. So we can refer to the colimit of a direct system.

Rmk: Can describe limit as nat. trf. to constant functor.

Thm 2.2.49 Let R be any ring. Every direct system of left R-mods over any part. ordered set has a limit.

Pf: See [Grifo Thm 1.76].

Many constructions arise as (co-) limits:

Def 2.2.50 Let $\mathcal{C}$ be a cat. A pullback of the arrows f and g consists of an object P and arrows p_1 and p_2 s.t.

$$\begin{tikzcd} P \arrow[r, dashed, "p_1"] \arrow[d, dashed, "p_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

commutes and satisfies univ. property:

$\forall$ objects Q and arrows q_1, q_2 s.t.

$$\begin{tikzcd} Q \arrow[r, "q_1"] \arrow[d, "q_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

commutes,

there exists a unique u s.t.

$$\begin{tikzcd} Q \arrow[dr, dashed, "u"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & \\ & P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\ & B \arrow[r, "g"'] & C \end{tikzcd}$$

commutes.

We say that

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

is a pullback diagram $\underbrace{}_{\text{limit}}$

Dually: Let $\mathcal{C}$ be a cat. A pushout of the arrows f, g

consists of object P and arrows p_1 and p_2 s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, dashed, "p_1"] \\ C \arrow[r, dashed, "p_2"'] & P \end{tikzcd} \circlearrowright \quad \text{and univ. property}$$

If
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "q_1"] \\ C \arrow[r, "q_2"'] & Q \end{tikzcd}$$
commutes, $\exists! u$ s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p_1"] \arrow[ddr, bend left, "q_1"] & \\ C \arrow[r, "p_2"] \arrow[drr, bend right, "q_2"'] & P \arrow[dr, dashed, "\exists! u"] & \\ & & Q \end{tikzcd}$$

commutes.

We say that
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p_1"] \\ C \arrow[r, "p_2"'] & P \end{tikzcd}$$
is a <u>pushout diagram</u>.

colimit

<u>Adjoint functors</u>

<u>Def</u> 2.2.57 Let $\mathscr{C}$ and $\mathscr{D}$ be locally small categories. Two covariant functors $\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$ form an <u>adjoint</u> pair (F, G) if given any objects $C \in \mathscr{C}$ and $D \in \mathscr{D}$, there is a bijection between the Hom sets

$$\mathrm{Hom}_{\mathscr{D}}(F(C), D) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(C, G(D))$$

which is natural on both objects, i.e., $\forall f \in \mathrm{Hom}_{\mathscr{C}}(C_1, C_2)$ and $g \in \mathrm{Hom}_{\mathscr{D}}(D_1, D_2)$ the diagrams

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{D}}(F(C_1), D) \arrow[r, "\cong"] \arrow[d, "F(f)^*"'] & \mathrm{Hom}_{\mathscr{C}}(C_1, G(D)) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathscr{D}}(F(C_2), D) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathscr{C}}(C_2, G(D)) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{D}}(F(C), D_1) \arrow[r, "\cong"] \arrow[d, "g_*"'] & \mathrm{Hom}_{\mathscr{C}}(C, G(D_1)) \arrow[d, "G(g)_*"] \\ \mathrm{Hom}_{\mathscr{D}}(F(C), D_2) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathscr{C}}(C, G(D_2)) \end{tikzcd}$$

commute $\forall C \in \mathcal{C}, D \in \mathcal{D}$. We say that F is <u>left adjoint</u> to G, or F has a <u>right adjoint</u>, and that G is the <u>right adjoint</u> of F, or that G has a <u>left adjoint</u>.

<u>Examples 2.2.52</u>

(1) ← Most important ex. in hom. alg.! (Hom-Tensor adjointness)
Let R, S be rings and M an R-S-bimodule (i.e. M left R-mod. and right S-module s.t. $\forall r \in R, s \in S, m \in M$: $(r \cdot m) \cdot s = r \cdot (m \cdot s)$)
Then $$\mathrm{Hom}_R(M \otimes_S X, Y) \cong \mathrm{Hom}_S(X, \mathrm{Hom}_R(M, Y))$$
for X S-module, Y R-module. So $(M \otimes_S -, \mathrm{Hom}_R(M, -))$ is an adjoint pair between S-mods and R-mods

$F: S\text{-Mod} \to R\text{-Mod}$, $X \mapsto M \otimes_S X$ $\quad \mathcal{C} = S\text{-mod}$, $\mathcal{D} = R\text{-mod}$

$G: R\text{-Mod} \to S\text{-Mod}$, $Y \mapsto \mathrm{Hom}_R(M, Y)$

$\mathrm{Hom}_R(M, Y)$ has left S-mod structure: $(s \cdot f)(m) = f(ms)$ (!)

(2) Fix a ring R. Consider the functor
$$\underline{\mathrm{Free}} : \underline{\mathrm{Set}} \to R\text{-Mod}$$
$$I \mapsto R^I = \bigoplus_I R$$

$\underline{\mathrm{Set}} \underset{\mathrm{Forget}}{\overset{\mathrm{Free}}{\rightleftarrows}} R\text{-mod}$

sending a set I to the free module indexed by I.
This functor is a left adjoint to the forgetful functor $R\text{-Mod} \to \underline{\mathrm{Set}}$ i.e., there is a natural bijection
$$\mathrm{Hom}_{R\text{-Mod}}(\underline{\mathrm{Free}}(I), M) \cong \mathrm{Hom}_{\underline{\mathrm{Set}}}(I, M)$$ identify image of forget with M

(in general, one can define a free module on a set I by the univ. property: given a function f from a set I to an R-module M, $\exists!$ R-module hom $\bigoplus_I R \to M$ that agrees with f on basis elements)

<u>Exercise</u>: Left and right adjoints are unique up to nat. isom.

§2.3. The category of chain complexes

Finally we can introduce the cat of chain complexes, and talk about exact seq. and homology (and the snake lemma, of course!)

Maps of Chain complexes

⇝ category, but need right notion for maps between cx.s ⇝ want homology to be functorial.

Def 2.3.1 Let $(F_\bullet, \partial_\bullet^F)$ and $(G, \partial_\bullet^G)$ be two complexes. A map of complexes or a chain map, written $h: (F_\bullet, \partial_\bullet^F) \to (G, \partial_\bullet^G)$ or simply $h: F \to G$, is a sequence of homomorphisms between R-modules $h_n: F_n \to G_n$ s.t. the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_{n+1} & \longrightarrow & F_n & \longrightarrow & F_{n-1} & \longrightarrow & \cdots \\
& & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n-1}} & & \\
\cdots & \longrightarrow & G_{n+1} & \longrightarrow & G_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots
\end{array}$$

This means: $h_n \partial_{n+1}^F = \partial_{n+1}^G h_{n+1} \quad \forall n.$

Note here: we say "map" although we mean R-module hom. here!

Ex 2.3.2 The 0 and id-maps $(F_\bullet, \partial_\bullet) \to (F_\bullet, \partial_\bullet)$ are:

$0_F: F_n \to F_n, \; x \mapsto 0$ and $id_F: F_n \to F_n, \; x \mapsto x \quad \forall n.$

Def 2.3.3 Let R be a ring. The category of chain complexes of R-modules, denoted Ch(R-Mod) or simply Ch(R) has objects: all cx. of R-modules and arrows: all maps of cx's of R-modules.

When $R = \mathbb{Z}$, write Ch(Ab) for Ch($\mathbb{Z}$), the cat of chain cx's of abelian groups.

Note that id-map above are precisely the identity arrows in $Ch(R)$.

Exercise Show that the isomorphisms in the cat $Ch(R)$ are precisely the maps of cx.s

$$\begin{tikzcd} \arrow[r] & F_{n+1} \arrow[r, "\partial^F_{n+1}"] \arrow[d, "h_{n+1}"'] & F_n \arrow[r] \arrow[d, "h_n"] & F_{n-1} \arrow[d, "h_{n-1}"] \\ & G_{n+1} \arrow[r, "\partial^G_{n+1}"'] & G_n \arrow[r] & G_{n-1} \end{tikzcd}$$

s.t. h_n is an iso $\forall n$.

$\Leftarrow$: If all h_n are isos: $\exists!\, u_n: G_n \to F_n$ s.t. $u_n \circ h_n = 1_{F_n}$, $h_n \circ u_n = 1_{G_n}$

The $\{u_n\}$ are chain maps: $u_n \circ \partial^G_{n+1} \overset{!}{=} \partial^F_{n+1} \circ u_{n+1}$

Know: $h_n \circ \partial^F_{n+1} = \partial^G_{n+1} \circ h_{n+1}$:

$$u_n \circ \partial^G_{n+1} = u_n \circ \partial^G_{n+1} \circ \underbrace{h_{n+1} \circ u_{n+1}}_{1_{G_{n+1}}} = \underbrace{u_n \circ h_n}_{1_{F_n}} \circ \partial^F_{n+1} \circ u_{n+1} = \partial^F_{n+1} \circ u_{n+1} \quad ✓$$

Clearly $\{u_n\}$ and $\{h_n\}$ are inverse to each other $\Rightarrow$ iso in $Ch(R)$.

$\Rightarrow$: If $\{u_n\}, \{h_n\}$ are inverse chain maps then $h_n \circ u_n = 1_{G_n}$, $u_n \circ h_n = 1_{F_n}$ by def.

by def all u_n, h_n are R-module homs $\Rightarrow$ isom. of R-mods.

This is a good notion of maps of cxs: it induces homomorphisms in homology, which allows us to say that homology is a functor.

Lemma 2.3.4 Let $h: (F_\bullet, \partial^F_\bullet) \to (G_\bullet, \partial^G_\bullet)$ be a map of cx. For all n, h_n restricts to homom. $B_n(h): B_n(F_\bullet) \to B_n(G_\bullet)$ and $Z_n(h): Z_n(F_\bullet) \to Z_n(G_\bullet)$
$\Rightarrow$ It induces homom. on homology $H_n(h): H_n(F_\bullet) \to H_n(G_\bullet)$.

Pf: Since $h_n \circ \partial^F_{n+1} = \partial^G_{n+1} \circ h_{n+1}$, any element $a \in B_n(F_\bullet)$, say $a = \partial^F_{n+1}(b)$ is taken to

$$h_n(a) = h_n \circ \partial^F_{n+1}(b) = \partial^G_{n+1}(h_{n+1}(b)) \in \operatorname{im}(\partial^G_{n+1}) = B_n(G_\bullet).$$

Similarly, if $a \in Z_n(F_\bullet) = \ker \partial^F_n$, then

$$\partial_n^{G} h_n(a) = h_{n-1}\, \partial_n^{F}(a) = h_{n-1}(0) = 0.$$

$\Rightarrow h_n(a) \in \ker(\partial_n^{G}) = Z_n(G_\bullet)$. Finally, the restriction of h_n to $Z_n(F_\bullet) \to Z_n(G_\bullet)$ sends $B_n(F_\bullet)$ into $B_n(G_\bullet)$, and thus it induces a well-def. homom. of the quotients $H_n(F_\bullet) \to H_n(G_\bullet)$.