WEEK 11

Last week: start of homological algebra part.
Categorical basics: some parts not discussed in lecture
These are greyed out.

<u>Example</u> 2.29: (1) In <u>Grp</u>, <u>Ring</u> and R-<u>Mod</u> the isos are the morphisms that are bijective fcts.
(2) In <u>Top</u> isos are homeom. which are bij. fct with continuous inverses.

<u>Exercises</u> (3) Show that in any cat, isomorphisms are both mono + epi.
Let f be iso and $A \underset{g_2}{\overset{g_1}{\rightrightarrows}} B \overset{f}{\underset{\exists g}{\rightleftarrows}} C$ comm., i.e. $fg_1 = fg_2$. $\exists g: C \to B$ s.t. $gf = 1_B$: $\Rightarrow g(fg_1) = g(fg_2)$
$\overset{\text{assoc}}{\Longleftrightarrow} \underbrace{(gf)}_{1_B} g_1 = \underbrace{(gf)}_{1_B} g_2$
$\overset{\text{id}}{\Longleftrightarrow} 1_B \circ g_1 = 1_B \circ g_2$
$\Longleftrightarrow g_1 = g_2 \Rightarrow f$ mono

Similar for epi.

(4) Show that the usual inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epi in the cat <u>Ring</u>.
Let $\mathbb{Z} \xrightarrow{\text{in}} \mathbb{Q} \underset{g_2}{\overset{g_1}{\rightrightarrows}} R$ with $g_1 \circ \text{inc} = g_2 \circ \text{inc} \Rightarrow g_1(i(1)) = g_1(1) = g_2(1)$. Any ring hom $\mathbb{Q} \to R$ is uniquely de[termined] by image of 1: $g(a/b) = a/b \cdot g(1)$ ~~etc~~ $\Rightarrow g_1 = g_2$
$\Rightarrow f$ epi.

$\Rightarrow$ epi $\neq$ surjective!

(5) Show that the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is mono in the cat of divisible abelian groups.

(6) In the poset cat. $\mathcal{P}$ every morphism is both monic + epic, but no nonidentity morphism has a left or right inverse.

We will need some special objects

Def 2.2.10 Let $\mathcal{C}$ be a cat. An initial object in $\mathcal{C}$ is an object i such that for every object $x \in ob(\mathcal{C})$, $Hom_{\mathcal{C}}(i,x)$ is a singleton, i.e. $\exists!$ arrow $i \to x$. A terminal object in $\mathcal{C}$ is a $t \in ob(\mathcal{C})$ s.t. for every object x in $\mathcal{C}$, $Hom_{\mathcal{C}}(x,t)$ is a singleton, i.e. $\exists!$ arrow $x \to t$. A zero object is an object that is both initial and terminal.

Exercise 7 Initial (terminal) objects are unique upto unique iso. $\leadsto$ Talk about "the" initial, terminal, zero object (if they exist!).

Ex 2.2.11 (1) $\emptyset$ is initial in Set. Any singleton is terminal. Since $\emptyset$ and $\{x\}$ are not isom. in Set, there is no 0-object in Set.

$M \to \{x\}$

singleton

(2) The 0 module is the 0 object in R-Mod.

(3) The trivial group $\{e\}$ is the zero object in Grp.

(4) In Rng, $\mathbb{Z}$ is the initial object, but there is no terminal object (unless we allow the 0 ring).

(5) There are no initial/terminal objects in cat of fields.

Now some obvious generalizations from Modules $\to$ Cats:

Def 2.2.12 A subcategory $\mathcal{C}$ of a cat $\mathcal{D}$ consists of a subcollection of the objects of $\mathcal{D}$ and subcoll. of morphisms of $\mathcal{D}$ s.t.

(a) For every object $C \in ob(\mathcal{C})$, the arrow $1_C \in Hom_{\mathcal{D}}(C,C)$ is an arrow in $\mathcal{C}$.

(b) For every arrow in $\mathcal{C}$, its source and target in $\mathcal{D}$ are objects in $\mathcal{C}$.

(c) For every pair of arrows f and g in $\mathcal{C}$, s.t. fg is an arrow

that makes sense in $\mathscr{D}$, fg is an arrow in $\mathscr{C}$.
In part, $\mathscr{C}$ is a cat.
A subcat $\mathscr{C}$ in $\mathscr{D}$ is a full subcat if $\mathscr{C}$ includes all of the arrows in $\mathscr{D}$ between any two objects in $\mathscr{C}$. $(\mathrm{Hom}_{\mathscr{C}}(C,D) = \mathrm{Hom}_{\mathscr{D}}(C,D)\ \forall C,D \in \mathscr{C})$

Ex 2.2.13 (a) R-mod - cat of f.g. R-mods is full subcat of R-Mod
(b) $\underline{\mathbf{Ab}}$ (ab. grp.) $\subseteq \underline{\mathbf{Grp}}$ full
(c) $\underline{\mathbf{Grp}}$ is subcat of $\underline{\mathbf{Set}}$, but not full. (not every maps betw. sets is a group hom!)

Def 2.2.14 Let $\mathscr{C}$ be a cat. The opposite cat $\mathscr{C}^{op}$ is the cat with the same objects as $\mathscr{C}$, but each arrow $f \in \mathrm{Hom}_{\mathscr{C}^{op}}(A,B)$ is the same as some arrow in $\mathrm{Hom}_{\mathscr{C}}(B,A)$. Composition fg in $\mathscr{C}^{op}$ is def. as comp. gf in $\mathscr{C}$.

→ "Dual notions": consider notion of $\mathscr{C}$ in $\mathscr{C}^{op}$: in practice: flip arrows!
e.g.: source ⟷ target, epi ⟷ mono, initial obj ⟷ terminal obj, homology ⟷ cohomology

Functors "functorial": construction behaves well w.r.t. morphisms

Def 2.2.15 Let $\mathscr{C}$ and $\mathscr{D}$ be cats. A covariant functor $F:\mathscr{C}\to\mathscr{D}$ is a mapping that assigns to each object A of $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$, and to each arrow $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$ an arrow $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(A),F(B))$ such that: (i) F preserves composition $F(fg) = F(f)F(g)$ for all composable arrows $f,g \in \mathscr{C}$.
(ii) F preserves the identity arrows: $F(1_A) = 1_{F(A)}\ \forall A \in \underline{ob}(\mathscr{C})$

Dually: a contravariant functor $F:\mathscr{C}\to\mathscr{D}$ is a mapping that assigns to each $A \in \underline{ob}(\mathscr{C})$ an $F(A) \in \underline{ob}(\mathscr{D})$ and $\forall f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$ an arrow $F(f) \in \mathrm{Hom}_{\mathscr{D}}(\underbrace{F(B),F(A)}_{\text{change of target + source!}})$ s.t.

(i') $\mathcal{F}$ preserves composition: $\mathcal{F}(fg) = \mathcal{F}(g)\mathcal{F}(f)$ for all composable arrows f, g in $\mathcal{C}$. $A \xrightarrow{g} B \xrightarrow{f} C$, $\mathcal{F}(A) \xleftarrow{\mathcal{F}(g)} \mathcal{F}(B) \xleftarrow{\mathcal{F}(f)} \mathcal{F}(C)$

(ii') $\mathcal{F}$ preserves identity arrows: $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$ $\forall A \in \underline{ob}(\mathcal{C})$.

Alternatively: a contravariant functor is a covariant functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}^{op}$.

Rmk 2.2.16 May also define contrav. functor $\mathcal{F}$ as cov. functor $\mathcal{F}: \mathcal{C}^{op} \to \mathcal{D}$

May also compose functors in obvious way: $\mathcal{F}: \mathcal{C} \to \mathcal{D}$, $\mathcal{G}: \mathcal{D} \to \mathcal{E}$ $\rightsquigarrow$ $\mathcal{G}\mathcal{F}: \mathcal{C} \to \mathcal{E}$

cov + cov $\Rightarrow$ comp. is covariant
contrav. + contrav. $\Rightarrow$ comp. is covariant (!)
$\mathcal{F}$ cov + $\mathcal{G}$ contrav $\Rightarrow$ $\mathcal{G}\mathcal{F}$ is contravariant

$$\begin{array}{ccccc} A & \overset{\mathcal{F}}{\rightsquigarrow} & \mathcal{F}(B) & \overset{\mathcal{G}}{\rightsquigarrow} & \mathcal{G}\mathcal{F}(A) \\ \downarrow & & \downarrow & & \downarrow \\ B & & \mathcal{F}(A) & & \mathcal{G}\mathcal{F}(B) \end{array}$$

Exercise Show that functors preserve isomorphisms. Let $f \in \underset{\mathcal{C}}{(A,B)}$ iso, $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ cov. functor. Then $\exists g \in \underset{\mathcal{C}}{(B,A)}$ s.t. $gf = 1_A$, $fg = 1_B$.

But $\mathcal{F}(gf) \overset{(i)}{=} \mathcal{F}(g)\mathcal{F}(f)$ and $\mathcal{F}(gf) = \mathcal{F}(1_A) \overset{(ii)}{=} 1_{\mathcal{F}(A)}$

Similarly: $\mathcal{F}(f)\mathcal{F}(g) = 1_{\mathcal{F}(B)}$ $\Rightarrow$ $\mathcal{F}(f): \mathcal{F}(A) \to \mathcal{F}(B)$ is iso.

Similar for $\mathcal{F}$ contravar.

Any functor sends isos to isos (since it preserves compositions + ids).

Examples 2.2.17 (a) For many cats: forgetful functors: forget higher structures and maps to $\underline{Set}$. E.g. $\underline{Grp} \to \underline{Set}$, $\underline{k\text{-Alg}} \to \underline{k\text{-Mod}}$.

(b) Identity functor $1_\mathcal{C}: \mathcal{C} \to \mathcal{C}$ and $f \mapsto f$ and constant functor at $C \in \underline{ob}(\mathcal{C})$: $\Delta C: \mathcal{C} \to \mathcal{C}$: $\Delta C(A) = C$ $\forall A \in \underline{ob}(\mathcal{C})$ and $f \in \underset{\mathcal{C}}{\mathrm{Hom}}(A,B)$ $\Delta C(f) = 1_C$.

need later (nat. trf)

(c) Unit group functor: $*: \underline{Ring} \to \underline{Grp}$ sends a ring R to its group of units R^*. (!) $*$ is a functor: objects ✓ morphisms: if $f: R \to S \in \underset{Ring}{\mathrm{Hom}}(R,S)$ then for any $u \in R^*$, we have $f(u \cdot u^{-1}) = f(u)f(u^{-1}) = f(1_R) = 1_S \Rightarrow f(u) \in S^*$. Thus f induces a function $R^* \to S^*$ by restricting f to R^*, which must be a group homom. since f preserves products.

(d) Given a ring homom $f: R \to S$, restriction defines a (faithful) functor $S\text{-mod} \to R\text{-mod}$. It's full $\Leftrightarrow$ f is an ring epi. (!)

(e) If M is an R-S bimodule, then any homom of S-modules $X \to X'$ gives a homom. $M \otimes_S X \to M \otimes_S X'$ of R-modules. Thus

$N\otimes_S -$: S-Mod $\to$ R-Mod becomes a functor.

(f) For a vector space V, dualization : $V \to V^*$ and $g \in \mathrm{Hom}_{K\text{-Vect}}(V,W) \to (g^* : W^* \to V^*)$ is a contravariant functor from K-Vect $\to$ K-Vect.

(g) Localization is a functor : Let R be a (comm.) ring and W be a multiplicatively closed set in R. Localization at W induces a functor R-mod $\to W^{-1}R$-mod : $M \to W^{-1}M$ and any $f \in \mathrm{Hom}_R(M,N)$ is sent to $W^{-1}f : W^{-1}M \to W^{-1}N$.

• A functor from KQ-mod $\to$ K-Mod is the same as a repr. of the quiver Q.

Rmk 2.2.18 If we apply functors to diagrams, we get the following :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] \\ C \arrow[r, "v"] & D \end{tikzcd} \quad \overset{F \text{ covariant}}{\rightsquigarrow} \quad \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "F(u)"'] & F(B) \arrow[d, "F(g)"] \\ F(C) \arrow[r, "F(v)"] & F(D) \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] \\ C \arrow[r, "v"'] & D \end{tikzcd} \quad \overset{F \text{ contravariant}}{\rightsquigarrow} \quad \begin{tikzcd} F(A) & F(B) \arrow[l, "F(f)"'] \\ F(C) \arrow[u, "F(u)"] & F(D) \arrow[l, "F(v)"'] \arrow[u, "F(g)"'] \end{tikzcd}$$

Def : The cat Cat has as objects all small cats and arrows all functors between them.

Def 2.2.19 A covariant functor $F : \mathcal{C} \to \mathcal{D}$ between locally small cats is : • faithful if all functions of sets

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(A),F(B)) : f \mapsto F(f) \quad (*)$$

are injective.

• full if all functions on sets $(*)$ are surjective.

• fully faithful if it is full and faithful.

• essentially surjective if every object D in $\mathcal{D}$ is isomorphic to $F(C)$ for some $C \in \mathrm{ob}(\mathcal{C})$. this is sometimes called "dense"

• an embedding if it is fully faithful and injective on objects.

Ex 2.2.20 : The forgetful functor R-Mod $\to$ Set is faithful

since any two maps with the same source and target coincide if and only if they are the same function of sets.
It is not full since not every map between the underlying sets of two R-modules is an R-module homom.
Also see Ex. 2.2.18 (d).

<u>Rmk 2.2.21</u> A fully faithful functor is not necessarily injective on objects, but it is injective on objects up to isomorphism.
A subcat $\mathcal{C}$ of $\mathcal{D}$ is full $\iff$ the inclusion functor $\mathcal{C} \to \mathcal{D}$ is full.

<u>Exercise</u>: Show that every fully faithful functor $F: \mathcal{C} \to \mathcal{D}$ respects isos:
(a) If f is an arrow in $\mathcal{C}$, s.t. $F(f)$ is an iso in $\mathcal{D}$, then f is an iso
(b) If $F(X)$ and $F(Y)$ are isom. objects in $\mathcal{D}$, then $X \cong Y$ in $\mathcal{C}$.

Finally, the two most important functors in homological algebra:
<u>Def 2.2.22</u> Let $\mathcal{C}$ be a locally small cat. An object A in $\mathcal{C}$ induces two <u>Hom-functors</u>:
- The <u>covariant $\mathrm{Hom}_{\mathcal{C}}(A,-)$</u> $: \mathcal{C} \to \underline{\mathrm{Set}}$ is defined as:

  on objects: $X \longmapsto \mathrm{Hom}_{\mathcal{C}}(A,X)$

  on arrows: $(B \xrightarrow{f} C) \longmapsto \left(\mathrm{Hom}_{\mathcal{C}}(A,B) \xrightarrow{f_* := \mathrm{Hom}_{\mathcal{C}}(A,f)} \mathrm{Hom}_{\mathcal{C}}(A,D)\right)$

  $g \longmapsto f \circ g$

  $\begin{array}{ccc} A & \xrightarrow{g} & B \\ & \searrow & \downarrow f \\ & & C \end{array}$ $f_*(g) = f \circ g$

We say that the functor $\mathrm{Hom}_{\mathcal{C}}(A,-)$ is <u>represented by A</u>

- The <u>contravariant functor</u> $\mathrm{Hom}_{\mathcal{C}}(-,B) \to \underline{\mathrm{Set}}$ is def. dually
  - on objects: $X \longmapsto \mathrm{Hom}_{\mathcal{C}}(X,B)$
  - on arrows: $(A \xrightarrow{f} C) \longmapsto \left(\mathrm{Hom}_{\mathcal{C}}(C,B) \xrightarrow{f^* := \mathrm{Hom}_{\mathcal{C}}(f,B)} \mathrm{Hom}_{\mathcal{C}}(A,B)\right)$

    $g \longmapsto g \circ f$

  $\begin{array}{ccc} A & \xrightarrow{f} & C \\ & \searrow & \swarrow g \\ & B & \end{array}$

Again $\mathrm{Hom}_{\mathcal{C}}(-,A)$ is the contravariant functor represented by A.
<u>Exercise</u>: Check that these two are indeed functors (!)

Natural transformations

Def 2.2.23 Let $F, G : \mathcal{C} \to \mathcal{D}$ be covariant functors. A <u>natural transformation</u> between F and G is a mapping that to each object A in $\mathcal{C}$ assigns an arrow $\eta_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), G(A))$ s.t. for all $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"] & G(B) \end{tikzcd}$$

commutes. $G \circ \eta = \eta \circ F$

[Sometimes written: $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{D} \end{tikzcd}$ or simply $\eta : F \Rightarrow G$].

Similarly, if F, G are contravariant functors from $\mathcal{C} \to \mathcal{D}$. Then a <u>natural transformation</u> between F and G: $\forall A \in \mathrm{ob}(\mathcal{C})$ assigns an arrow $\eta_A : \mathrm{Hom}_{\mathcal{D}}(F(A), G(A))$ s.t. $\forall f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$:

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] & G(A) \\ F(B) \arrow[r, "\eta_B"] \arrow[u, "F(f)"] & G(B) \arrow[u, "G(f)"'] \end{tikzcd}$$

commutes. $G \circ \eta = \eta \circ F$

η is called <u>natural isomorphism</u> if each η_A is an isomorphism.

<u>Example 2.2.24</u> The determinant gives rise to a nat. trf: Fix $n \geq 1$ and consider the functor

$$GL_n : \underline{Ring} \to \underline{Grp} \quad \leftarrow \text{group (!)}$$
$$R \mapsto GL_n(R) = \{A \in M_n(R) \text{ invertible}\}$$
$$\underset{\text{ring hom}}{(f : R \to S)} \mapsto GL_n(f) : GL_n(R) \to GL_n(S)$$
$$A = (a_{ij}) \longmapsto \underline{f(A)} = (f(a_{ij}))_{ij}$$

apply f to all entries

this is group hom (!)

<u>Claim</u>: det is a natural trf. from the GL_n functor to the unit functor $(\ )^*$ from Ex. 2.2.17(c).

<u>Pf</u>:
$$\begin{tikzcd} GL_n(R) \arrow[r, "\det"] \arrow[d, "f"'] & R^* \arrow[d, "f"] \\ GL_n(S) \arrow[r, "\det"] & S^* \end{tikzcd}$$

First note that $\det(A) \in R^*$ for any invertible matrix A, so we have

a map $\det : GL_n(R) \to R^*$. Moreover for any given ring map $f: R \to S$, the diagram above commutes:
$\det \circ f(A) = \underbrace{f}_{f \text{ restricted to } R^*} \circ \det(A)$ since ring hom!

Note:
We identify f with both the map $GL_n(f)$ ← apply f to all entries of A and $f|_{R^*}$.
So comm. diag just encodes the fact that taking dets commutes with ring homomorphisms. □

Ex 2.2.25

(1) Any morphism $y: X \to Y$ in a cat $\mathcal{C}$ defines a natural trf of (representable) functors $\Phi : \mathrm{Hom}_{\mathcal{C}}(Y,-) \to \mathrm{Hom}_{\mathcal{C}}(X,-)$ with
$\Phi_Z : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Y) : f \mapsto f \circ y$.
$X \xrightarrow{y} Y \xrightarrow{f} Z$

(2) If k is a field and V a k-vectorspace, there is a natural map $V \to V^{**} : v \mapsto (\overset{: V \to k}{f} \mapsto f(v))$. This is a nat. trf. $1_{\mathcal{C}} \to (-)^{**}$ of functors from k-Mod to k-Mod. If $\dim_k V < \infty$ (i.e. use cat. k-mod), then get nat. iso. (!)

$1_{k\text{-mod}}$: $F : k\text{-Mod} \to k\text{-mod}$
$V \mapsto V$
$f \in \mathrm{Hom}(V,W) \mapsto f$

$(-)^{**}$: $G : \mathcal{C} \to \mathcal{C}$
$V \mapsto V^{**}$
$f \in \mathrm{Hom}(V,W) \mapsto f^{**} : f \in \mathrm{Hom}(V,W)$

Let $V \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(V,W)$

$$\begin{tikzcd} W & V \arrow[l, "f"'] \arrow[d] \\ & k \arrow[ul, leftarrow, "g"] \end{tikzcd}$$

$$\begin{tikzcd} 1_{\mathcal{C}}(V) \arrow[r, "\eta_V"] \arrow[d, "f"] & V^{**} \arrow[d, "f^{**}"] \\ 1_{\mathcal{C}}(W) \arrow[r, "\eta_W"] & W^{**} \end{tikzcd}$$

$\eta_V \in \mathrm{Hom}_{\mathcal{C}}(V, V^{**})$
$\eta_V(v) = (v \mapsto (f \mapsto f(v)))$ $\quad \forall g \in \mathrm{Hom}(W,k)$

commutes: $\eta_W \circ f(v) = (f(v) \mapsto (g \mapsto gf(v)))$
$f^{**} \circ \eta_V(v) = f^{**}(v \mapsto f(v)) \overset{?}{=} (f(v) \mapsto (g \mapsto gf(v)))$

$f: V \to W$
$f^*: W^* \to V^*$, $g \mapsto gf$
$f^{**}: V^{**} \to W^{**}$
$(g \mapsto gf) \mapsto$

(3) A map of R-S-bimodules $M \to N$ gives nat. trfs
(i) $\mathrm{Hom}_R(N,-) \to \mathrm{Hom}_R(M,-)$ of functors R-Mod $\to$ S-Mod
(ii) $\mathrm{Hom}_R(-,M) \to \mathrm{Hom}_R(-,N)$ — " — $(R^{op}\text{-Mod}) \to (S^{op}\text{-Mod})$
(iii) $M \otimes_S - \to N \otimes_S -$ — " — $(S\text{-Mod}) \to R\text{-Mod}$ etc

Def 2.2.26 Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between the cats $\mathcal{C}$ and $\mathcal{D}$. We write
$$\mathrm{Nat}(F,G) := \{\text{natural transformations } F \Rightarrow G\}.$$
From this one can build a <u>functor category</u> which objects are all covariant functors $\mathcal{C} \to \mathcal{D}$ and arrows correspond to nat. transf. This cat is denoted by $\mathcal{D}^{\mathcal{C}}$ (or $\mathrm{Fun}(\mathcal{C},\mathcal{D})$)

For the functor cat to be a "true" cat, we need to compose nat. trfs:

Consider 2 nat. trfs:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathcal{D} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd}$$

We can compose them to form a nat. trf:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\eta\varphi"] & \mathcal{D} \end{tikzcd}$$

For each object C in $\mathcal{C}$, $\eta\varphi(C)$ = the arrow $F(C) \xrightarrow{\varphi_C} G(C) \xrightarrow{\eta_C} H(C)$.

This makes the diagram:

$$\begin{tikzcd} F(A) \arrow[r, "\varphi_A"] \arrow[d, "F(f)"'] \arrow[rr, bend left, "\eta_A\varphi_A"] & G(A) \arrow[r, "\eta_A"] \arrow[d, "G(f)"] & H(A) \arrow[d, "H(f)"] \\ F(B) \arrow[r, "\varphi_B"] \arrow[rr, bend right, "\eta_B\varphi_B"'] & G(B) \arrow[r, "\eta_B"] & H(B) \end{tikzcd}$$

commute: replace horizontal arrows with compositions $\eta\varphi$
$\rightsquigarrow$ comm. diagram.

Def 2.2.27 Two cats $\mathcal{C}$ and $\mathcal{D}$ are <u>equivalent</u> if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphism $\alpha : GF \Rightarrow 1_{\mathcal{C}}$ and $\beta : FG \Rightarrow 1_{\mathcal{D}}$. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an <u>equivalence</u> if there exist a functor G anon nat. isos α and β as above.

Assuming the axiom of choice, one can show

<u>Thm 2.2.28</u> F is an equivalence if and only if F is fully faithful and essentially surjective.

<u>Pf</u>: $\Leftarrow$ optional

$\Rightarrow$: Suppose $\exists G : \mathcal{D} \to \mathcal{C}$ and since nat. trf. nat. isos: $\alpha : GF \Rightarrow 1_{\mathcal{C}}$, $\beta : FG \Rightarrow 1_{\mathcal{D}}$.
For $\theta \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ we get $\theta\alpha_X = \alpha_Y G(F(\theta))$. So if $F(\theta) = F(\theta')$, then $\theta\alpha_X = \theta'\alpha_X \Rightarrow \theta = \theta'$, since α_X is an iso. Thus F is faithful.
Similarly: G is faithful.
Suppose $\varphi \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$. Let $\theta = \alpha_Y G(\varphi)\alpha_X^{-1} \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$
Then $\theta\alpha_X = \alpha_Y G(F(\theta))$ gives $G(\varphi) = G(F(\theta)) \Rightarrow \varphi = F(\theta)$, and thus F is full.
Also, any $Y \in \underline{ob}(\mathcal{D})$ is isomorphic to $F(G(Y))$, so F is ess. surj.

$\Leftarrow$ uses axiom of choice

$\Leftarrow$: Let F be fully faithful + ess. surj. Then $\forall Z \in \underline{ob}(\mathcal{D})$ choose $G(Z) \in \underline{ob}(\mathcal{C})$ and an iso $\eta_Z : Z \to F(G(Z))$. We extend it to a functor $G : \mathcal{D} \to \mathcal{C}$ by defining $G(\theta)$ for $\theta \in \mathrm{Hom}_{\mathcal{D}}(Z,W)$ to be the unique morphism $g \in \mathrm{Hom}_{\mathcal{C}}(G(Z), G(W))$ with $F(g) = \eta_W \theta \eta_Z^{-1}$. □

<u>Examples 2.2.29</u>

(a) If K is a field, then there is an equivalence of cats from the cat with objects $\mathbb{N}$ and $\mathrm{Hom}(m,n) = M_{n\times m}(K)$ to the cat of K-mod of fin. dim'l K-vector spaces, sending n to K^n and a matrix A to the corr. linear map.

(b) The following are equiv. for a quiver Q:
 (1) KQ-mod.
 (2) The cat of K-representations of Q.
 [(3) The functor cat from the path cat of Q to K-mod.]

(c) Let $\mathcal{C}$ be the cat with one object C and a unique arrow 1_C. Let $\mathcal{D}$ be the cat with two objects D_1, D_2 and 4 arrows: $1_{D_1}, 1_{D_2}$, and two isos $\alpha : D_1 \to D_2$, $\beta : D_2 \to D_1$. Let $\mathcal{E}$ be the cat with objects E_1, E_2 and 2 arrows $1_{E_1}, 1_{E_2}$.

Show: (i) $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

(ii) $\mathcal{C}$ and $\mathcal{E}$ are not equivalent.

Use Thm: $F: \mathcal{C} \to \mathcal{D}$, $C \mapsto D_1$, $F(1_C) = 1_{D_1}$

$G: \mathcal{D} \to \mathcal{C}$, $D_i \mapsto C$, $1, \alpha, \beta \mapsto 1_C$.

F, G functors: ✓ (Def ✓) $F(\underbrace{\alpha\beta}_{1}) = F(1)F(1)$ ✓

$FG(D_i) = D_1$ $\quad$ $\mathcal{D} \xrightarrow{FG} \mathcal{D}$

$$\begin{array}{ccc} FG(D_1) & \xrightarrow{1} & D_1 \\ \downarrow{\scriptstyle FG(\alpha)} & & \downarrow{\scriptstyle \alpha} \\ FG(D_2) & \xrightarrow{\alpha} & D_2 \end{array} \checkmark \qquad \begin{array}{c} FG(D_2) \\ \downarrow{\scriptstyle FG(\beta)} \\ FG(D_1) \end{array}$$

etc.

With thm: • F ess. surj.: $D_1 = F(C)$ $\quad D_2 \cong D_1 = F(C)$ ✓

• F full/faithful: $\mathrm{Hom}_{\mathcal{C}}(C,C) \to \mathrm{Hom}_{\mathcal{D}}(F(C),F(C)) = \mathrm{Hom}_{\mathcal{D}}(D_1,D_1)$

$\{1_C\} \overset{F}{=} \{1_{D_1}\}$

$\Rightarrow$ inj + surj ✓

Def 2.2.30 A covariant functor $F: \mathcal{C} \to \underline{Set}$ is representable if $\exists$ an object A in $\mathcal{C}$ s.t. F is naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(A,-)$.

A contravariant functor $F: \mathcal{C} \to \underline{Set}$ is representable if $\exists$ object $B \in \mathcal{C}$ s.t. F is nat. isomorphic to $\mathrm{Hom}_{\mathcal{C}}(-,B)$.

Ex 2.2.31 $id: \underline{Set} \to \underline{Set}$ is representable. (!)

Forgetful functor R-mod $\to \underline{Set}$ is representable

Optional:

The Yoneda Lemma (= "Most important statement in cat. theory")

Thm 2.2.32 (Yoneda Lemma): Let $\mathcal{C}$ be a locally small cat and fix an object A in $\mathcal{C}$. Let $F: \mathcal{C} \to \underline{Set}$ be a covariant functor. Then there is a bijection

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(A,-), F) \xrightarrow{\gamma} F(A).$$

Moreover, this correspondence is natural in both A and F.

Pf: See e.g. [Hilton-Stammbach, Prop II.4.1]

Rmk: (1) In particular, this means that the collection of all nat. trfs from $Hom_{\mathcal{C}}(A,-)$ to F is a set. (not clear a priori, since $\underline{ob}(\mathcal{C})$ is not necessarily a set!)

(2) If we apply the Yoneda lemma to the case when F is itself a Hom-functor, $F = Hom_{\mathcal{C}}(B,-)$, the Yoneda lemma says that there is a bijection between

$Nat(Hom_{\mathcal{C}}(A,-), Hom_{\mathcal{C}}(B,-))$ and $Hom_{\mathcal{C}}(B,A)$.

In part., each arrow in $\mathcal{C}$ determines a nat. trf between Hom-functors.

Yoneda embedding:

Let $\mathcal{C}$ be a locally small cat. Each arrow $f: A \to B$ in $\mathcal{C}$ gives rise to a nat. trf. $Hom_{\mathcal{C}}(-,A) \Rightarrow Hom_{\mathcal{C}}(-,B)$ that send each object X to the arrow

$$Hom_{\mathcal{C}}(X,A) \xrightarrow{f_*} Hom_{\mathcal{C}}(X,B)$$
$$g \longmapsto fg$$

See this in diag: for each arrow $g: X \to Y$ we have

$$\begin{tikzcd} X \arrow[d, "g"] & Hom_{\mathcal{C}}(X,A) \arrow[r, "f_*"] & Hom_{\mathcal{C}}(X,B) \\ Y & Hom_{\mathcal{C}}(Y,A) \arrow[r, "f_*"] \arrow[u, "Hom(g,A) = g^*"] & Hom_{\mathcal{C}}(Y,B) \arrow[u, "Hom_{\mathcal{C}}(g,B) = g^*"'] \end{tikzcd}
\qquad
\begin{tikzcd} & Y \arrow[dl, "h"'] \arrow[d, "\alpha"] & X \arrow[l, "g"'] \arrow[dl, dotted, "\alpha \circ g = g^*(\alpha)"] \\ A \arrow[r, "f"] & B \arrow[dl] & \\ {} & & \end{tikzcd}$$

This diag commutes, since $g^* \circ f_*(h) = g^*(fh) = (fh) \circ g = f(\overbrace{h\, g}^{X \to A}) =$ ($h: Y \to A$)

$= f_*(hg) = f_* g^*(h)$.

Conversely, f^* indicates the nat. trf $Hom_{\mathcal{C}}(B,-) \Rightarrow Hom_{\mathcal{C}}(A,-)$ sending each object X in $\mathcal{C}$ to the arrow

$$Hom_{\mathcal{C}}(B,X) \xrightarrow{f^*} Hom_{\mathcal{C}}(A,X)$$
$$g \longmapsto gf.$$

Thm 2.2.33 (Yoneda embedding) Let $\mathcal{C}$ be a locally small cat. The covariant functor

$$\begin{array}{ccc} \mathcal{C} & \to & \underline{Set}^{\mathcal{C}^{opp}} \\ A & & Hom_{\mathcal{C}}(-,A) \\ \downarrow f & \longmapsto & \downarrow f_* \\ B & & Hom_{\mathcal{C}}(-,B) \end{array}$$

from $\mathcal{C}$ to the cat. of contravariant functors $\mathcal{C} \to \underline{Set}$ is an embedding. Moreover, the contravariant functor

$$\begin{array}{ccc} \mathcal{C} & \to & \underline{Set}^{\mathcal{C}} \\ A & & Hom_{\mathcal{C}}(A,-) \\ \downarrow f & \longmapsto & \uparrow \\ B & & Hom_{\mathcal{C}}(B,-) \end{array}$$

from the cat $\mathcal{C}$ to cat of covariant functor $\mathcal{C} \to \underline{Set}$ is also an embedding.

Pf: Left out. [See e.g. [Hilton-Stammbach, Cor II.4.2]]

Thm 2.2.34: Let $X, Y \in \underline{ob}(\mathcal{C})$ of a loc. small cat $\mathcal{C}$. If $Hom_{\mathcal{C}}(-,X)$ and $Hom_{\mathcal{C}}(-,Y)$ are nat. isomorphic, or if $Hom_{\mathcal{C}}(X,-)$ and $Hom_{\mathcal{C}}(Y,-)$ are nat. isomorphic, then X and Y are isomorphic objects.

Pf: Left out. [See lecture notes of [Grifo])

Summary of Yoneda Lemma:

(1) To give a nat. trf from $Hom_{\mathcal{C}}(A,-)$ to $\mathcal{F}$ is the same as giving an elt. in the set $\mathcal{F}(A)$.

(2) The collection of all natural trfs from $Hom_{\mathcal{C}}(A,-)$ to $\mathcal{F}$ is a set.

(3) To give a nat. trf between repres. functors is to give an arrow between the corr. representing objects.

(4) Every loc. small cat $\mathcal{C}$ can be embedded in $\underline{Set}^{\mathcal{C}^{(op)}}$ ⇝ Study functor cat $\underline{Set}^{\mathcal{C}}$

(5) We can recover an object in $\mathcal{C}$ by knowing maps to and from it.

Products and Coproducts

Def 2.2.35 Let $\mathcal{C}$ be a loc. small cat and consider a family of objects $\{A_i\}_{i\in I}$ in $\mathcal{C}$. The <u>product</u> of the A_i is an object in $\mathcal{C}$ denoted by $\prod_{i\in I} A_i$, together with arrows $\pi_j \in \mathrm{Hom}_{\mathcal{C}}(\prod_j A_j, A_j)$ $\forall j \in I$, the <u>projections</u> satisfying the following universal property: given any object B in $\mathcal{C}$ and arrows $f_i : B \to A_i$ $\forall i$, there exists a unique arrow f such that

$$\begin{tikzcd} B \arrow[d, dotted, "\exists! f"'] \arrow[dr, "f_j"] & \\ \prod_j A_j \arrow[r, "\pi_j"'] & A_j \end{tikzcd}$$

commutes $\forall j$