§2.1 Intro + Motivation

Homological algebra first appeared in connection to topological spaces. "Homology" is a way of assoc. a sequence of modules (or groups, or ..) to another object, e.g., a topological space.
⇝ homology of top. space encodes all top. info about the space in algebraic language.

More formally: Idea is to study complexes and their homology from an abstract perspective.

Some of the most important definitions: let R be any (unital) ring

Def 2.1.1 A (chain) complex of R-modules $(C_\bullet, \partial^\bullet)$ is a sequence of R-modules C_i and R-module homom. need: $\mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0 \ \forall n$. We say that C_n is the module in homological degree n, and the maps ∂_n are the differentials of the complex.
(Sometimes: omit ∂_n and just write $C_\bullet$ or $\partial_\bullet$ for cx.)
[There is also a co-chain complex where arrows go the other way

$$\cdots C_{n-1} \xrightarrow{\partial_n} C_n \xrightarrow{\partial_{n+1}} C_{n+1} \cdots$$ ⇝ not used here]

We say a complex C is bounded above if $C_n = 0 \ \forall n \gg 0$ and bounded below if $C_n = 0 \ \forall n \ll 0$. A bounded cx is one that is bounded above and below. If $C_\bullet$ is bounded, we just write: $C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_m$.

Rmk 2.1.2: $\partial_n \circ \partial_{n+1} = 0 \Rightarrow \mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$.

Def 2.1.3 The cx. $(C_\bullet, \partial_\bullet)$ is exact at n if $\ker \partial_{n+1} = \mathrm{im}\, \partial_n$. An exact seq. is a cx. that is exact everywhere:

$$M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \longrightarrow \quad \text{s.t. } \ker f_n = \mathrm{im}\, f_{n+1}$$

Already seen: <u>short exact seq (ses)</u> : $0 \to A \to B \to C \to 0$.

<u>Rmk 2.1.4</u> 1st isom. thm : $0 \to A \to B \to C \to 0$ ses $\Rightarrow C \cong B/\underset{\cong A}{f(A)}$

<u>Notation: 2.1.5</u>: $A \twoheadrightarrow B$ denotes a surjective map
$A \hookrightarrow B$ denotes an injective map

<u>Def 2.1.6</u>: The <u>cokernel</u> of $A \xrightarrow{f} B$ is $\mathrm{coker}(f) := B/\mathrm{im}(f)$.

$\leadsto$ For ses $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$: $A = \ker g$, $C = \mathrm{coker} f$.

<u>Ex 2.1.7</u> (1) ses of $\mathbb{Z}$-mods: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$

(2) Let $I \subseteq R$ be an ideal, $i: I \hookrightarrow R$ inclusion, $\pi: R \to R/I$ proj.
$\Rightarrow 0 \to I \xrightarrow{i} R \xrightarrow{\pi} R/I \to 0 \to 0$ ses

(3) Let $R = k[x]/(x^2)$, then have ∞-exact cx:
$$\cdots \to R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \to \cdots \qquad \begin{pmatrix} \mathrm{im}(\cdot x) = Rx \\ \ker(\cdot x) = Rx \end{pmatrix}$$

(4) $0 \to M \xrightarrow{f} N \to 0$ is exact $\Leftrightarrow$ f is isomorphism.

(5) $0 \to M \to 0$ is exact $\Leftrightarrow$ $M = 0$.

These chain cx. first appeared in alg. top $\leadsto$ homology.

<u>Def 2.1.8</u>: The <u>homology</u> of the complex $(C_\bullet, \partial_\bullet)$ is the sequence of R-modules $H_n(C_\bullet) = H_n(C) := \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$.

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$

The <u>n-th homology</u> of $(C_\bullet, \partial_\bullet)$ is $H_n(C)$. The submodules $Z_n(C_\bullet) = Z_n(C) := \ker(\partial_n) \subseteq C_n$ are called <u>cycles</u>, and the submodules $B_n(C_\bullet) = B_n(C) := \mathrm{im}(\partial_{n+1}) \subseteq C_n$ are called <u>boundaries</u>.

Sometimes: an element $x \in B_n(C)$ is called <u>boundary</u> (n-boundary), and $x \in Z_n(C)$ is called <u>cycle</u> (n-cycle).

The homology of a complex measures how far the cx. is from being exact at each step. Again: say cohomology of cochain cx., written $H^n(C^\bullet) = H^n(C) = \ker(\partial_{n+1}) / \operatorname{im}(\partial_n)$ ← won't be working with those.

Rmk 2.1.9 A cx. $(C_\bullet, \partial_\bullet)$ is exact at $n \iff H_n(C_\bullet) = 0$.

Example 2.1.10 (1) Let $R = k[x]/(x^3)$ and consider the cx:

$$C_\bullet = \quad \cdots \to R \xrightarrow{\cdot x^2} R \xrightarrow{\cdot x^2} R \xrightarrow{\cdot x^2} \cdots$$

Here $\partial_n(\alpha) = \alpha \cdot x^2$ $\forall n$ and $\operatorname{im}(\partial_n) = \langle x^2 \rangle$ ← the ideal $\langle x^2 \rangle \subseteq R$

$\ker(\partial_n) = \{\alpha \in R : \alpha x^2 = 0\} = \langle x \rangle$ ← clearly $\ker(\partial_n) \supseteq \langle x \rangle$; if α has const. term $\Rightarrow \alpha \cdot x^2$ has term of deg 2 $\Rightarrow \neq 0$.

Thus $\forall n$: $H_n(C_\bullet) = (x)/(x^2) \cong R/(x) \cong k$.

(2) Let $C_\bullet: \cdots 0 \to \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0 \cdots$ cx. of ab. groups

$n = 2 \quad 1 \quad 0$

Here $\partial_2(m) = 4m$, $\partial_1(m) = \pi(m) = m \bmod 2$ and $\partial_n(m) = 0$ $\forall n \in \mathbb{Z} \setminus \{2, 1\}$

$\Rightarrow$ • $H_n(C) = 0$ $\forall n \geq 3$ and $\forall n \leq -1$

- $H_2(C) = \ker(\cdot 4) / \operatorname{im}(0) = 0/0 = 0$.
- $H_1(C) = \ker(\pi) / \operatorname{im}(\cdot 4) = 2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$
- $H_0(C) = \ker(\mathbb{Z}/2 \to 0) / \operatorname{im}(\pi) = (\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) = 0$.

$C_\bullet$ is exact everywhere, except at $n = 1$.

In order to develop more concepts in homological algebra, we need some category theory language - introduce some basics, then get back to hom. algebra.

§2.2. Cat. theory basics

"Abstract nonsense": language to talk about objects with the same properties: e.g. rings + homom. ⟷ groups + homom. ⟷ top. spaces + cont. maps

Intros to cat. theory: "collection" of obj. in cat, not "set"

In order to not run into problems, we use ZFC (Zermelo-Fraenkel with choice) axiomatization of set theory - find details elsewhere!

Categories

Def 2.2.1 A category $\mathcal{C}$ consists of the following data:

- a collection of objects: $ob(\mathcal{C})$,
- for each two objects $A, B \in ob(\mathcal{C})$, a collection $Hom_{\mathcal{C}}(A,B)$ of arrows or morphism from A to B, and
- for each three objects $A, B, C \in ob(\mathcal{C})$, a composition

$$Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) \longrightarrow Hom_{\mathcal{C}}(A,C)$$
$$(f, g) \longmapsto g \circ f$$
↳ mostly write gf,

satisfying the following axioms:

(1) The $Hom_{\mathcal{C}}(A,B)$ are all disjoint. In part., if $f \in \mathcal{C}$ is an arrow, then its source is A and its target is B as objects $A, B \in \mathcal{C} \rightsquigarrow f \in Hom_{\mathcal{C}}(A,B)$.

(2) For each $A \in ob(\mathcal{C})$, there is an identity arrow $1_A \in Hom_{\mathcal{C}}(A,A)$ s.t. $1_A \circ f = f$ and $g \circ 1_A = g$ $\forall f \in Hom_{\mathcal{C}}(B,A)$ and $\forall g \in Hom_{\mathcal{C}}(A,B)$.

(3) Composition is associative: $f \circ (g \circ h) = (f \circ g) \circ h$ for all appropriately chosen arrows.

Notation: Write: $f: A \to B$ or $A \xrightarrow{f} B$ for $f \in Hom_{\mathcal{C}}(A,B)$.

Ex 1: Show that every element in a cat. has a unique identity morphism. (Let $A \xrightarrow{1_A} A$ and $A \xrightarrow{1_A'} A$ be id arrows

BOLD Set
↳ write underline.

Example 2.2.2: (1) The cat Set with objects all sets and arrows all functions between sets.

(2) **Grp** whose objects are the collection of all groups and arrows = group homom.. arrows 1_G are identity homom.

(3) **Ab** with objects abelian groups and arrows are homom. of abelian groups, identity = id. homom.

(4) **Ring** with object rings and arrows ring homom.

(5) **R-Mod** ← R-mod in 2N! left modules over a fixed ring R with R-module homom.
(Mod-R : right modules)

Similar R-mod : f.g. R-modules and their homoms.
If $R = k$ a field, then objects are k-vector spaces and arrows linear trfs. ⇝ write **Vect k**.

(6) The cat **Top** of top. spaces and continuous fcts.

One may consider many variations, e.g: fin.dim'l vs. (n-dim'l vs) with all lin. trf.

Not cat: (n-dim'l) vs. + nonzero lin. maps : not closed under $\circ$
— " — + maps with det = 0 : $\nexists$ identity maps

(7) **Set*** : (pointed sets) : object (X, x) sets X with pt $x \in X$ and morphisms $(X, x) \xrightarrow{f} (Y, y)$ are $f: X \to Y$ s.t. $f(x) = y$ + compos.

(8) The empty cat has no objects and no arrows.

(9) Path cat. of a quiver Q: objects $= Q_0$ and $\mathrm{Hom}(i, j)$ = set of paths from i to j. (k-lin. path cat of Q: obj. $= Q_0$, $\mathrm{Hom}(i,j) = k$-module with basis the paths $i \to j$)

Def 2.2.3 A cat $\mathcal{C}$ is called *locally small* if for all objects A, B in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set. A cat $\mathcal{C}$ is *small* if it is locally small and the collection of objects is a set. In loc. small cat: Hom-sets.

Concrete cat: objects are sets with extra structure and arrows are fct.s between objects.

E.g: **Set** is locally small but not small.

Ex. 2.2.4 (need for limits) Let $(X, \leq)$ be a partially ordered set: X itself is a cat:

objects are $x \in X$, and $\forall x,y \in X$: $\mathrm{Hom}_X(x,y) = \begin{cases} \text{singleton} & \text{if } x \leq y \text{ or} \\ \emptyset & \text{if } x \not\leq y \end{cases}$

Only one way to form composition: if there is an arrow $i \to j$ and an arrow $j \to k$, then $i \leq j$ and $i \leq k \Rightarrow \exists$ unique arrow $i \to k$.
This cat is clearly locally small, since all non-empty Hom-sets are singletons. It is also small, since the objects are (by construction) the set X. This cat. is called <u>poset category</u> denoted by <u>PO</u>(X).
→ ex: $\underline{n}$ = cat with objects $0,1,\dots,n-1$ and $\mathrm{Hom}(i,j) = \begin{cases} \emptyset & \text{if } i > j \\ \text{singleton} & \text{if } i \leq j \end{cases}$

This is <u>PO</u>$(\{0,\dots,n-1\}, \leq)$

<u>Rmk 2.2.5</u> A locally small cat with just one element is completely determined by its unique Hom-set: it thus consists of a set S with an associative operation that has an identity element ⇝ we call this a <u>semi-group</u>.

Now we get to morphisms: most concepts in cat. theory can be understood via diagrams ⇝ "hom. algebra is a way to study commutative diagrams."

<u>Def 2.2.6</u>. A <u>diagram</u> in a category $\mathcal{C}$ is a directed multigraph whose vertices are objects in $\mathcal{C}$ and whose arrows are morphisms in $\mathcal{C}$. A <u>commut. diagram</u> in $\mathcal{C}$ is a diagram in $\mathcal{C}$ in which for each pair of vertices A and B, any two paths from A to B compose to the same morphism.

<u>Ex 2.2.7</u> The diagram
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "g"] \\ D \arrow[r, "v"'] & C \end{tikzcd}$$
<u>commutes</u> $\Leftrightarrow g \circ f = vu$.

<u>Def 2.2.8</u> Let $\mathcal{C}$ be any category.
(1) an arrow $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is <u>left (right) invertible</u> if there

exists $g \in \mathrm{Hom}_{\mathcal{C}}(B,A)$ such that $gf = 1_A$ $(fg = 1_B)$
In this case, g is called the *left (right) inverse of f*.
(g is left (right) inverse of f if the following diag commutes:

left:
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "id_A"'] & B \arrow[d, "g"] \\ & A \end{tikzcd} \qquad gf = 1_A$$

right:
$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "1_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd} \qquad fg = 1_B$$
)

(2) An arrow $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is an *isomorphism* if $\exists\, g \in \mathrm{Hom}_{\mathcal{C}}(B,A)$ s.t. $gf = 1_A$ and $fg = 1_B$. Then g is the *inverse* of f. Two objects A and B are *isomorphic* if $\exists$ an isomorphism $A \to B$.

(3) An arrow $f \in \mathrm{Hom}_{\mathcal{C}}(B,C)$ is *monic* (or *monomorphism* or *mono*) if for all arrows $g_1, g_2 : A \to B$: $A \overset{g_1}{\underset{g_2}{\rightrightarrows}} B \xrightarrow{f} C$ if $fg_1 = fg_2$, then $g_1 = g_2$

(4) An arrow $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is an *epi* or (*epimorphism* or *epic*) if for all arrows $g_1, g_2 : B \to C$: $A \xrightarrow{f} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} C$ if $g_1 f = g_2 f$, then $g_1 = g_2$.

Exercise: Show that in *Set* the monos coincide with inj. fcts and epis coincide with surjective fcts.