①

WEEK 9

<u>Recall</u>: We are in the middle of the Proof of

Theorem 1.6.17 (Gabriel) (1) If I is an admissible ideal in KQ, then $R = KQ/I$ is finite dimensional K-algebra, $J(R) = (KQ)_+/I$ and $R/J(R) \cong K \times \dots \times K$.
(2) Conversely, if R is a finite dim'l K-algebra and $R/J(R) \cong K \times \dots \times K$ then $R \cong KQ/I$ for some finite quiver Q and some admissible ideal I.

Pf: (1) ✓

(2) Idea: Last time.

More detailed: Let $J = J(R)$ be the Jacobson radical of R.
By our assumption $\underset{S}{\underset{\shortparallel}{R/J}} \cong \underbrace{K \times \dots \times K}_{n\text{-factors}}$, so S has a basis of orthogonal idempotents $f_i = (0, \dots, 0, 1, 0, \dots, 0)$, $i \in \{1, \dots, n\}$.
Set $f := \sum_{i=1}^{n-1} f_i = (1, \dots, 1, 0)$, then
$$fSf = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} = \left(\begin{array}{ccc|c} 1 & & & 0 \\ & \ddots & & \\ & & 1 & 0 \\ \hline 0 & & & 0 \end{array}\right)$$
$$= \left(\begin{array}{c|c} \mathbb{1}_{n-1} & 0 \\ \hline 0 & 0 \end{array}\right) \cong K^{n-1} = \{(a_1, \dots, a_{n-1}, 0)\}$$

Now we show by induction on n that if $R/J \cong K^n$, then there are orthogonal idempotents e_i in R lifting the idempotents f_i in R/J.
Let e be a lift of f, i.e., $e \in R$ idempotent, such that $f = e + J \in R/J$ and $f - e \in J$. Such a lift exists for any idempotent of a K-alg. $A/J(A)$, where A is a f. dim'l K-algebra. ([see ASS, Lemma I.4.4. p19]:
Let $\eta = g + J(A) \in A/J(A)$, $g \in A$, be an idempotent, then $\exists$ idemp. $e \in A$, s.t. $e - g \in J(A)$: <u>Pf</u>: We know $J(A)^m = 0$ for some $m > 1$. Since $\eta^2 = \eta$, we have $(g + J(A))(g + J(A)) = g^2 + J(A) = g + J(A)$ and thus $(g^2 - g) \in J(A)$.
$$\Rightarrow (g - g^2)^m = 0 = g^m - g^{m+1} \underbrace{\left(\sum_{j=1}^{m} (-1)^{j-1} \binom{m}{j} g^{j-1} \right)}_{=: r}$$

$\overset{(i)}{\Rightarrow} g^m = g^{m+1}\cdot t$ and $\overset{(ii)}{g}t = tg$ (computation!)

Claim: $e := (gt)^m$ is the idempotent lifting n.

Pf Claim: $e = g^m t^m \overset{(i)}{=} (g^{m+1}t)t^m = g^{m+1}t^{m+1} = \ldots = g^{2m}t^{2m} = ((gt)^m)^2 = e^2$

so e is idempotent.

Next: $\overset{(iii)}{g - g^m} \in J(A)$ because $g - g^2 \in J(A)$ implies:

$g - g^m = g(1-g^{m-1}) = g(1-g)(1+\ldots+g^{m-2}) = (g-g^2)(1+\ldots+g^{m-2}) \in J(A)$ ✓

Further: $g - gt \in J(A)$ because

$g + J(A) \overset{(iii)}{=} g^m + J(A) \overset{(i)}{=} g^{m+1}t + J(A) = (g^{m+1} + J(A))(t + J(A)) \overset{iii}{=}$
$(g^m + J(A))(g + J(A))(t + J(A)) \overset{iii}{=} (g + J(A))(g + J(A))(t + J(A)) = (g^2 + J(A))(t + J(A))$
$\overset{g^2 - g \in J}{=} (g + J(A))(t + J(A)) = gt + J(A)$. ✓

Now: $e + J(A) = g^m t^m + J(A) \overset{(ii)}{=} (gt + J(A))^m = (g + J(A))^m = g^m + J(A) = g + J(A)$

and $\Rightarrow e - g \in J(A)$.] $\square$

If e is the lift of f in R, then eJe is a nilpotent ideal in eRe.

[$(eJe)^2 = eJeeJe = eJeJe \overset{J \text{ two-sided id.}}{\subseteq} eJ^2e \overset{\text{induction}}{\rightsquigarrow} (eJe)^m \subseteq eJ^m e = 0$.]

$\overset{\text{Lemma 1.4.6}}{\Longrightarrow} eJe \subseteq J(eRe)$

$eRe/eJe \overset{!}{\cong} e\underbrace{(R/J)}_{S}e \overset{e \text{ lifts } f}{=} f(R/J)f \cong K^{n-1}$. ← Use again [ASS I.1.4]

$\in (M/N) \cdot eM/eN$

Thus $J(eRe) = eJe$, and by induction there are idempotents $e_1, \ldots, e_{n-1}$ in eRe inducing the idempotents $f_1, \ldots, f_{n-1}$ in R/J. Then take $e_n = 1 - e$.

Now construct Q_R:

Set $(Q_R)_0 = \{1, \ldots, n\}$. We have $J \overset{\text{Peirce decomp}}{=} \bigoplus_{i,j} e_j J e_i$. Then J/J^2 is both

a left and a right R-module [an R-R-bimodule], so decompose

as $J/J^2 = \bigoplus_{i,j \in Q_0} e_j (J/J^2) e_i = \bigoplus_{i,j} (e_j J e_i)/(e_j J^2 e_i)$.

Set now the arrows $i \to j$ in Q_R correspond to elements of $e_j J e_i$ inducing a K-basis of $(e_j J e_i)/(e_j J^2 e_i)$.

We get a homomorphism $\Theta: KQ \to R$.

Claim: Θ is surjective ($\overset{\text{isom. thm}}{\Rightarrow} R \cong KQ/I$, where $J^m \subseteq I \subseteq J$ for some $m \geq 1$)

and $\ker\theta = I \subseteq KQ$ is an admissible ideal.

J/J^2 = basis of U

Pf of claim: Let $U = \theta(KQ_+) \subseteq J$. Since $U \subseteq J \Rightarrow U + J^2 = J$
$\xRightarrow{\text{Nakayama lemma}}$ $U = J$.

This implies that $\theta : KQ \to R$ is surjective. □claim

Let now $I := \ker\theta$. Still have to show: I is admissible.
If $m >> 0$ such that $J^m = 0$, then $\underbrace{\theta((KQ_+)^m)}_{\theta \text{ alg. hom}} = \theta(KQ_+)^m \overset{U = \theta(KQ_+)}{=} U^m = 0$

$\Rightarrow (KQ_+)^m \subseteq \ker\theta = I$.

For the other inclusion ($I \subseteq KQ_+^2$), let $\underbrace{x \in I}_{\theta(x) = 0}$. We may write

$$x = \underbrace{u}_{K\text{-lin. comb of } e_i\text{'s}} + \underbrace{\overset{\text{arrs}}{v}}_{K\text{-lin. comb of some arrows} \in Q} + \overset{\in KQ_+^2}{w} \in KQ_+^2.$$

Since $\theta(e_i) = e_i$ by construction and $\theta(\overset{KQ_+}{v}), \theta(\overset{KQ_+}{w}) \in J$ ($\operatorname{Im}(KQ_+) = J$).

we must have $\theta(x) = \underbrace{\theta(u)}_{\notin J \text{ (or } 0)} + \underbrace{\theta(v) + \theta(w)}_{\in J} = 0 \Rightarrow u = 0$ (e_i K-lin. indep)

$\Rightarrow \theta(v) = -\underbrace{\theta(w)}_{\in J^2} \Rightarrow \theta(v) \in J^2$. This means that $\theta(v) + J^2 = 0$ in J/J^2.

But this means that $v = 0$ (arrows are the only $\neq 0$ elts in $KQ_+/(KQ_+)^2$)

$\Rightarrow x = w \in KQ_+^2$. Thus $I \subseteq (KQ_+)^2 \subseteq KQ_+$. □

Examples: (1) Let $A = K[x]/(x^m)$, $m \geq 2$. This is a fin. dim'l K-algebra (even comutative!) with $J(A) = (\bar{x}) = (x + (x)^m)$

- $A/J \cong K$, so we only have one idempotent e_1, so $Q_0 = \{1\}$.
- $J/J^2 = (\bar{x})/(\bar{x}^2) \cong K\bar{x}$ is a 1-dim'l K-ves. $\Rightarrow$ only one arrow

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[loop right, "\alpha"] \end{tikzcd} \qquad KQ_+ = \langle \alpha, \alpha^2, \dots, \alpha^{m-1} \rangle \qquad \begin{aligned} \theta : KQ &\longrightarrow K[x]/(x^m) \\ e_1 &\longmapsto 1 \\ \alpha &\longmapsto \bar{x} \end{aligned}$$

Then: $\Theta(KQ_+) = \langle \bar{x}, \bar{x}^2, \dots, \bar{x}^{m-1} \rangle \Rightarrow \ker\Theta = \langle \alpha^m \rangle$

$$\Rightarrow K[x]/(x^m) \cong K(\bullet \circlearrowleft^{\alpha})/(\alpha^m).$$

(2) $e^{ij}: j \to i$

Let $A = \begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & K & K & 0 \\ K & K & 0 & K \end{bmatrix}$ ← fin. dim. K-algebra

Can find 4 idempotents

$e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_2, e_3, e_4$

$\underbrace{A = \bigoplus_{i=1}^{4} Ae_i}$; so 4 idempotents are gen. A

Here $J(A) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & K & 0 & 0 \end{bmatrix}$ and $J^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ K & 0 & 0 & 0 \end{bmatrix}$

$J^3 = 0.$

Compute: $e_j J/J^2 e_i =$ $\begin{array}{c} e_jJe_1 \cdots e_jJe_4 \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & K & 0 & 0 \end{bmatrix} \end{array} \Big/ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ K & 0 & 0 & 0 \end{bmatrix}$

$i \to j$

K-basis: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & K & 0 & 0 \end{bmatrix}$ ⇝ get 3 arrows: $\alpha: 1 \to 2$
$\beta: 2 \to 3$
$\gamma: 2 \to 4$

$\dim(e_j A e_i) = 1$ or 0.

So: Q:
$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\ & & 4 \end{tikzcd}$$

$\Theta: KQ \to A$
$e_i \mapsto E_i$
$\alpha \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
etc.

See here that $\ker\Theta = 0$ (e.g. $\beta\alpha \neq 0 \Rightarrow \Theta(\beta\alpha) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in J^2$)
$\neq 0$

Rmks: • One can also relate quivers to K-algebras A of ∞-dimension ⇝ "Ext-quiver", very useful when interested in homological properties (gl.dim, projectives, $D^b(\mathrm{mod}\,A)$)

In particular, to any semi-perfect noetherian ring, one can associate a quiver, as we did in the f.dim'l case. See [Hazewinkel-Gubareni-Kirichenko, "Algebras, Rings and Modules", Vol. I] Chapter 11

- In rep. theory, one also associates different quivers to an algebra A, e.g. the Auslander-Reiten quiver (=AR-quiver) which encodes the structure of mod-A (i.e. irred. maps between A-modules).

also see hom. algebra!