Recall : $KQ$ = path algebra of quiver
modules over $KQ$ $\xleftrightarrow{1:1}$ representations of $Q$
$V \longmapsto (e_iV$ , maps $V_a : V_i \to V_j$ given as $a: i \to j$ for $a \in e_jVe_i$ )

( $V = \bigoplus_{i \in Q_0} V_i$ , with action $e_i v = v_i$ and $a_1 \cdots a_n v = V_{a_1} \circ V_{a_2} \circ \cdots V_n(v_{t(a_n)}) \in V_{h(a_1)}$ ) $\longleftrightarrow$ $(V_i , V_a : V_i \to V_j)$

---

Example for this correspondence :
(6) $Q$
$$\begin{tikzcd} 1 \arrow[dr] & & \\ & 3 & 4 \arrow[l] \\ 2 \arrow[ur] & & \end{tikzcd} \qquad V: \begin{tikzcd} K \arrow[dr, "1"] & & \\ & K & K \arrow[l, "1"'] \\ K \arrow[ur, "1"] & & \end{tikzcd} \qquad V' = \begin{tikzcd} K \arrow[dr, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & & \\ & K^2 & K \arrow[l, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"'] \\ K \arrow[ur, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] & & \end{tikzcd}$$

Then $\mathrm{Hom}_K(V', V)$ is 2-dim'l : a morphism $V' \to V$ is determ. by 5 scalars $a, b, c, d, e$ s.t.
$$\begin{tikzcd} K \arrow[dr, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[ddd, bend right, "{[a]}"'] & & \\ & K^2 \arrow[dd, "{[c\ d]}"] & K \arrow[l, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"'] \arrow[dd, "{[e]}"] \\ K \arrow[ur, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] \arrow[dd, bend right, "{[b]}"'] & & \\ K \arrow[r, "1"] & K & K \arrow[l, "1"'] \\ K \arrow[ur, "1"'] & & \end{tikzcd}$$
commutes

get : $c + d = e$ (left)
$c = a$ (upper)
$d = b$

So choice of $a, b$, determines the morphism completely $\Rightarrow K^2$.

More on the correspondence :

(2) Further $KQ$-submodules $W$ of $V$ correspond to tuples $(W_i)$ where each $W_i$ is a $K$-submodule of $V_i$, s.t. $V_a(W_i) \subseteq W_j$ for all arrows $a: i \to j$.
Then $W$ corresponds to the subrepresentation $(W_i , V_a|_{W_i} : W_i \to W_j)$ and $V/W$ to the representation $(V_i/W_i : \overline{V_a} : V_i/W_i \to V_j/W_j)$.

(3) Direct sums of modules $V = \bigoplus_{\lambda \in \Lambda} V^\lambda$ correspond to direct sums of representations $(\bigoplus_{\lambda \in \Lambda} V_i^\lambda , \bigoplus_{\lambda \in \Lambda} V_a^\lambda)$.

Now we are approaching a theorem of P. Gabriel, that essentially says that any f. dim'l $K$-algebra $A$ (with some restriction depending on $A/J(A)$) will be of the form $KQ/I$ ← admissible ideal

<u>Notation</u> 1.6.6: (a) Denote by $(KQ)_+$ the $K$-span of the nontrivial paths. This is an ideal (clear: mult increases length) and $(KQ)/(KQ)_+ \cong K^{(Q_0)}$. ← also clear $KQ/KQ_+ = \langle e_i, e_j \rangle$

(b) We write $P[i]$ for the $KQ$-module $KQe_i$ (paths starting at $i$), so $KQ = \bigoplus_{i \in Q_0} P[i]$. ← i.e. $P[i]$ are projective modules

Considered as a representation of $Q$, the vector space at vertex $j$ has as basis the paths from $i$ to $j$.

e.g. $Q: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ $\quad P[1]: K \xrightarrow{\alpha} K \xrightarrow{\beta} K$ $\quad P[3]: 0 \to 0 \to K$

$P[2]: 0 \to K \xrightarrow{\beta} K$

written as modules: $P[1] = {}_K\langle e_1, \alpha, \alpha\beta \rangle$ $\quad P[2] = {}_K\langle e_2, \beta \rangle$

$P[3] = {}_K\langle e_3 \rangle$ $\quad KQ \cong$ 6-dim'l $K$-algebra

$P[i]$ are the projectives of $KQ$.

(c) We write $S[i]$ for the representation with $S[i]_i = K$ and $S[i]_j = 0$ for $j \neq i$ and all $S[i]_\alpha = 0$. It corresponds to the module $KQe_i/(KQ)_+e_i$ ← we will see that these are the simple $KQ$-modules.

For the rest of this section we will assume that $K$ = field.

<u>Example</u>: $Q = 1 \to 2$ $\quad P[1]: K \xrightarrow{1} K$ $\quad S[1] = K \xrightarrow{0} 0$

$P[2] = S[2] = 0 \xrightarrow{0} K$

- Compute $\mathrm{Hom}(S[1], P[1]) = 0$ $\quad \mathrm{Hom}(S[2], P[1]) = K$.
- The subspaces $(K \le V_1, 0 \le V_2)$ does not give a subspace of $P[1]$. But $(0 \le V_1, K \le V_2)$ is a subrep of $P[1]$ and it is $\cong$ to $S[2]$.
- There is a s.e.s. $0 \to S[2] \to P[1] \to S[1] \to 0$.
- $S[1] \oplus S[2] \cong K \xrightarrow{0} K$ and for $\lambda \neq 0 \in K$ we have $K \xrightarrow{\lambda} K \cong P[1]$.
- Every representation of $Q$ is isomorphic to a direct sum of copies of $S[1]$, $P[1]$, and $S[2]$: For a f. dim'l rep. $V_1 \xrightarrow{a} V_2$ one can see

this as follows: take bases of $V_1, V_2$, then the rep. is isom. or to $K^n \xrightarrow{A} K^{m}$ $n$ for some $m\times n$ matrix $A$. ~ may write as $K^n \xrightarrow{A} K^n$ $n \geq m$ with Now Gauss-elim. tells us that $\exists P, Q$ invertible s.t. $PAQ^{-1} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, with $I$ = id. matrix.
Then get isom. of reps
$$\begin{tikzcd} K^n \arrow[r, "A"] \arrow[d, "Q"'] & K^m \arrow[d, "P"] \\ K^n \arrow[r, "PAQ^{-1}"] & K^{m}\ n \end{tikzcd}$$
, and the last rep. is a direct sum as claimed.

Lemma 1.6.7 (i) The $P[i]$ are non-isom. indec. $KQ$-modules
(ii) The $S[i]$ are non-isom. simple $KQ$-modules.
(iii) If $|Q_0| < \infty$ and contains no oriented cycles; ← i.e. $KQ$ is fin. dim'l $K$-algebra! then $(KQ_+)$ is nilpotent, Lemma 1.6.4! so it is the Jacobson radical of $KQ$ and the $S[i]$ are all of the simple modules.

Pf: (i) The spaces $P[i] = KQe_i$, $e_j KQ e_i$, and $e_j KQ$ have as $K$-bases the paths with tail at $i$, with tail at $i$ and head at $j$, and with head at $j$.
If $f \neq 0 \in KQe_i$ and $g \neq 0 \in e_i KQ$, then $fg \neq 0$: explicitly, if $p$ and $q$ are paths of max. length involved in $f$ and $g$, then the coefficient of $pq$ in $fg$ must be $\neq 0$. $eAe \cong \mathrm{End}_A(Ae)^{op}$
Now by $\boxed{\text{Lemma 2}}$: $\mathrm{End}_{KQ}(P[i])^{op} = \mathrm{End}_{KQ}(KQe_i, KQe_i)^{op} \cong e_i KQ e_i$ as rings and by the observation above this is a domain (products of $\neq 0$ elements are non-zero). This means that $e_i KQ e_i$ has no non-trivial idempotents $\overset{\text{Lemma 3}}{\Longrightarrow}$ $P[i]$ is indecomposable.
(assume $e \in A$ is a domain and has non-trivial idempotent $e \Rightarrow A = Ae \oplus A(1-e)$ then $\underset{\neq 0}{e} \in Ae$ and $(\underset{\neq 0}{1-e}) \in A(1-e) \Rightarrow \underset{\neq 0}{e}(\underset{\neq 0}{1-e}) = e - e^2 = 0$ $\Rightarrow A$ not dom.)
If $P[i] \cong P[j]$, then there are inverse isomorph. between the two modules, and so for an element $f \in e_j KQ e_i$, and element $g \in e_i KQ e_j$ we have $fg = e_j$ and $gf = e_i$. Then, by argument above $f$ and $g$ can only involve trivial paths, so $i = j$. (product of nontrivial paths cannot be trivial!)

(ii) Clear (there are no nonzero proper submodules of $K$!)

(iii) Since $|Q_0| < \infty$ and $Q$ contains no oriented cycles, there is a bound on the length of any path. So $(KQ)_+$ is nilpotent.

This means $\mathcal{J}(KQ) \supseteq (KQ)_+$.

Since $KQ/(KQ)_+ \cong K^{(Q_0)}$, it is semi-simple [direct product of simple module $K$]

This means $(KQ)_+ \supseteq \mathcal{J}(KQ)$. $\Rightarrow \mathcal{J}(KQ) = (KQ)_+$.

The simples are indexed by $Q_0$ ($\to KQ/(KQ)_+ \cong K^{Q_0}$), so there are no other simples than the $S[i]$. $\square$ Alter: [ASS I.1.4 (c): $A/I \cong K^n \Rightarrow I = \operatorname{rad} A$)

**Def 1.6.8**: Suppose that $|Q_0| < \infty$. An ideal $I \subseteq KQ$ is <u>admissible</u> if

(1) $I \subseteq (KQ)_+^2$ and

(2) $(KQ)_+^n \subseteq I$ for some $n$.

The pair $(Q, I)$ is the called a <u>bound quiver</u> and $KQ/I$ a <u>bound quiver algebra</u>.

<u>Examples 1.6.9</u>: (1) If $Q$ has no oriented cycles, then $I = 0$ is admissible ($0 \subseteq (KQ)_+^2$ and $(KQ)_+$ nilpotent by Lemma 1.6.7. (iii))

(2) Let $Q$ be the quiver $\overset{1}{\bullet} \circlearrowleft x$, then $KQ = K[x]$. The admissible ideals in $KQ$ are of the form $(x^n)$, $n \geq 2$. ($Q$ not acyclic, so $KQ_+ \neq \mathcal{J}(KQ)$ in this case)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0$

(3) Let $Q$ be the quiver

$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "b"'] & 2 \arrow[d, "c"] \\ 3 \arrow[r, "d"'] & 4 \end{tikzcd}$$

and let $I$ be the ideal $I = (ca - db)$

Here $KQ_+ = (a, b, c, d, ca, db)$

$\mathcal{J}(KQ)^2 = (ca, db)$

$\mathcal{J}(KQ)^3 = 0$

$\Rightarrow \underbrace{\mathcal{J}(KQ)^3}_{0} \subseteq I \subseteq \mathcal{J}(KQ)^2$.

and $I$ is admissible.

<u>Lemma 1.6.10</u> Let $Q$ be a finite quiver and $I \subseteq KQ$ be an ad=

missible ideal. Then the $(\varepsilon_i = e_i + I)_{i \in Q_0}$ form a complete ⑤
set of primitive orthogonal idempotents of $KQ/I$.

Pf: Since the $\varepsilon_i$ are the image of the orth. idempotents $e_i$ under the projection $KQ \to KQ/I$, complete + orth. follows.
For primitive note that te $e_i$ are primitive ($e_i$ prim $\Leftrightarrow e_i R e_i \cong \mathrm{End}_R(Re_i)$ local (*)) (Show $\varepsilon_i$ and $0$ are only non-triv. idempotent in $\varepsilon_i KQ/I \varepsilon_i$: If $\sigma$ is another idemp., then it is of the form $\sigma = \lambda e_i + w + I$, where $\lambda \in K$, $w \in \varepsilon_i KQ/I \varepsilon_i$ is a lin comb of cycles thr. $i$ of length $\geq 1$.

(*) criterion is proven in [Schiffler] cor. 4.13

Then $0 \overset{KQ/I}{=} \sigma^2 - \sigma = (\lambda e_i + w)(\lambda e_i + w) - (\lambda e_i + w) =$
$= \lambda^2 e_i + \lambda w + \lambda w + w^2 - \lambda e_i - w = (\lambda^2 - \lambda) e_i + (2\lambda - 1) w + w^2$
i.e. $(\lambda^2 - \lambda) e_i + (2\lambda - 1) w + w^2 \in I$. Since $I \subseteq KQ_+^2$: $\lambda^2 - \lambda = 0$, i.e. $\lambda = 0$ or $\lambda = 1$.
- $\lambda = 0$, then $\sigma = w + I$. But since $I \supseteq KQ_+^m \Rightarrow w^m \in I$, so $w^m = 0$ in $KQ/I$. so $w \in I$ and $\sigma = 0$.
- If $\lambda = 1$, then $\sigma = e_i + w + I$, so $e_i - \sigma = -w + I$ is also idempotent in $\varepsilon_i KQ/I \varepsilon_i$ $\Rightarrow w$ idemp. mod $I$ $\overset{\text{similar as before}}{\Longrightarrow}$ $e_i = \sigma$.

[see lemma II.2.4 in [ASS]] □

Prop 1.6.11: Let $Q$ be a finite quiver and $I$ an admissible ideal. Then $R = KQ/I$ is a finite dim'l $K$-algebra.

Pf: Since $I$ is admissible, $\exists m \geq 2$ s.t. $(KQ)_+^m \subseteq I$. But then there is a surj. algebra homom: $KQ/(KQ)_+^m \to KQ/I$. Thus it is enough to prove that $KQ/(KQ)_+^m$ is finite dim'l. Now the residue classes of paths of length $< m$ form a $K$-basis of $KQ/(KQ)_+^m$. There are only fin. many of these $\Rightarrow KQ/(KQ)_+^m$ is fin. dim'l.

Ex 1.6.12 (Dieudonné): If $I$ is not admissible, then $KQ/I$ may not be f. gen. Let $Q$: $\beta \circlearrowleft \cdot \circlearrowright \alpha$ and $I = \langle \beta^2, \alpha\beta \rangle$. $I$ not adm., because $\alpha^m \notin I$ for any $m \geq 1$, but $\alpha^m \in (KQ)_+^m = \langle \alpha^m, \beta^m, \alpha^{m-1}\beta, \dots \rangle$
$KQ/I$ is not fi. dim $\rightsquigarrow$ cf. ex. II.2.5 in [ASS].

Lemma 1.6.13 Let $KQ$ be a finite quiver. Then every admissible ideal $I$ is f.gen. In part. the generators are of the form $\{\rho_1,\dots,\rho_n\}$, where $\rho$ is a relation: $\rho = \sum_{i=1}^{m} \lambda_i w_i$, $\lambda_i \in K$, where $w_i$ are paths of length $\geq 2$ s.t. if $i \neq j$, then the source (resp. target) of $w_i$ coincides with that of $w_j$).

e.g.: 


Relation: $\rho = ba + fc$, $fed$

Pf: See Lemma II 2.8 + Cor II.2.9 of [ASS]. $\square$

Lemma 1.6.14 Let $Q$ be a finite quiver and $I$ be an admissible ideal $\subseteq KQ$ and let $R = KQ/I$. Then $J(R) = (KQ)_+/I$ and $R/J(R) \cong K^{|Q_0|}$.

Pf: Since $I$ is admissible, $\exists m \geq 2$, s.t. $(KQ)_+^m \subseteq I \Longrightarrow$ $((KQ)_+/I)^m = 0$ and $(KQ)_+/I$ is an nilpotent ideal in $R$.
On the other hand $\underbrace{(KQ/I)}_{R} / \underbrace{(KQ)_+/I}_{J(R)} \underset{\text{isom thm}}{\cong} KQ/(KQ)_+ \cong K^{|Q_0|}$. $\square$

$\Longrightarrow$ assertion (again Lemm I.1.4 [ASS])

Cor 1.6.15: For any $\ell \geq 1$ we have $J(KQ/I)^\ell = ((KQ)_+/I)^\ell$. $\square$

Rmk 1.6.16 Look at $J(KQ/I)/J(KQ/I)^2 \underset{\text{Lemma 1.6.14}}{\cong} (KQ)_+/I \,/\, ((KQ)_+/I)^2$

Clearly: $J(KQ)/I \,/\, J(KQ)^2/I \cong (KQ)_+/(KQ)_+^2$.

$\cong (KQ)_+/(KQ)_+^2$ as $K$-vs.

as a basis we have $\bar{\alpha} + J(KQ/I)^2$, where $\bar{\alpha} = \alpha + KQ/I$ and $\alpha \in Q_1$ is an arrow. $\square$

1.6.17 Theorem (Gabriel) (1) If $I$ is an admissible ideal in $KQ$, then $R = KQ/I$ is finite dimensional $K$-algebra, $J(R) = (KQ)_+/I$ and $R/J(R) \cong K \times \dots \times K$.

(2) Conversely, if $R$ is a finite dim'l $K$-algebra and $R/J(R) \cong K \times \dots \times K$ then $R \cong KQ/I$ for some finite quiver $Q$ and some admissible

ideal $I$.

Pf: (1) Follows from Prop 1.6.11 and Lemma 1.6.15.

(2) First note that for any idempotent $e \in R$ ← $k$-alg. and $M$ an $R$-module and $N \subseteq M$ submodule, we have $e(M/N) \underset{k\text{-mods.}}{\cong} eM/eN$.

We sketch this direction, complete argument in [ASS, II.3]: The idea is to construct a quiver $Q_R$ for $R$ and then show that $R \cong kQ_R/I$ as $k$-algebras. Construction: know that $R/J(R)$ is s.s. $\rightsquigarrow$ basis of orth. idempotents $\rightsquigarrow$ "lift" idempotents to $R$ (Wakeyama lemma!) → vertices of $Q_R$

arrows = "paths of length 1 in $kQ$"

$\rightsquigarrow$ so "$(kQ_+)/(kQ)_+^2$" $= J/J^2$