

Ex. 1.5.7 (Group algebras) Let K be a commutative ring and $(G, \cdot)$ be a (semi-)group. The (semi-)group algebra $K[G]$ (or KG) is a free K -module with basis $g \in G$ (sometimes written e_g) with mult:
$$\left(\sum_{g \in G} a_g e_g \right) \left(\sum_{g \in G} b_g e_g \right) := \sum_{g \in G} \left(\sum_{\substack{g_1, g_2 \in G \\ g_1 \cdot g_2 = g}} a_{g_1} b_{g_2} \right) e_g$$

The map $K \rightarrow KG, a \mapsto a e_1$ $\forall a \in K$ is a ring monom. and hence we may consider K as a subring of KG . Since $K \subset Z(KG)$,

$$a \in K: a \cdot \sum_{g \in G} a_g g = a \cdot 1_G \cdot \sum_{g \in G} a_g g = \sum_{g \in G} (a a_g) g = \sum_{g \in G} (a_g a) g = \left(\sum_{g \in G} a_g g \right) \cdot (a \cdot 1_G) = \sum_{g \in G} (a_g a) g \cdot 1_G$$
since $K = \text{comm.}$

KG is a K -algebra (see Rmk 1.5.2. (2)). The map $G \rightarrow KG, g \mapsto e_g$ is a (semi-)group monomorphism and hence we may consider G as a sub(semi-)group of KG .

Thm 1.5.8 (Maschke) Let K be a field and G be a finite group. Then KG is semi-simple $\Leftrightarrow \text{char } K \nmid |G|$.

Pf:

$\Rightarrow$: Assume that $p \mid |G|$. We show that $K[G]$ contains a non-zero nilpotent element. Then Thm 1.4.10 $\Rightarrow KG$ is not semisimple.

The element $z = \sum_{g \in G} 1 \cdot g = \sum_{g \in G} e_g$ is $\neq 0$ and since $hz = zh \ \forall h \in G$

$$hz = h \left(\sum_{g \in G} g \right) = \sum_{g \in G} hg = \sum_{g \in G} g = z$$

$$zh = \left(\sum_{g \in G} g \right) h = \sum_{g \in G} gh = \sum_{g \in G} g = z$$
since gh runs through G as g does.

We have $z \in Z(K[G])$. Thus $gz = z \forall g \in G$ (2)

$$z^2 = \left(\sum_{g \in G} g \right) z = \sum_{g \in G} (gz) = |G| \cdot z = \underbrace{\frac{|G|}{p}}_{\in \mathbb{N}} \cdot \underbrace{(pz)}_{= 0, \text{ since char } K = p} = 0.$$

$\square$.

$\Leftarrow$: See presentation. $\square$

Introduce: Indecomposable modules and idempotents for K -algebras ③

Let R be a commutative ring and let A be a K -algebra.

Recall that $e \in A$ is idempotent if $e^2 = e$; e_i, e_j are orthogonal if $e_i e_j = 0$ for $i \neq j$; and $e_1, \dots, e_n$ is complete if $1 = e_1 + \dots + e_n$.

Lemma 1 If M is a left A -module, then have to show that $eM = M$!!

(i) If e is idempotent, then $eM = \{em : em = m\}$ is a K -submodule of M . (usually not an A -submod! e.g. $A = M_2(K)$, $e_1 A = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$ is not left A -mod!)

(ii) If (e_i) are orthogonal idempotents, then the sum $\sum_{i \in I} e_i M$ is direct.

(iii) If $e_1, \dots, e_n$ is a complete family of orthogonal idempotents, then $M = e_1 M \oplus \dots \oplus e_n M$.

Pf: All straightforward, e.g. (i) if $em = m$, then $m \in eM$, while if $m \in eM$, then $m = em' = e^2 m' = e(em') = em$. K -submod clear! $\square$

Prop (Peirce decomposition) If $e_1, \dots, e_n$ is a complete family of orthogonal idempotents, then $A = \bigoplus_{i,j=1}^n e_i A e_j$.

Write as a matrix:

$$A = \begin{bmatrix} e_1 A e_1 & e_1 A e_2 & \dots & e_1 A e_n \\ \vdots & \ddots & \ddots & \vdots \\ e_n A e_1 & \dots & \dots & e_n A e_n \end{bmatrix} \quad \text{then mult. in } A \text{ over to matrix multiplication.}$$

Rmk If e is an idempotent, then eAe is an algebra with the same operations as A , with unit element e . Since the unit is not the same as for A , it is not an A -subalgebra. Sometimes eAe is called the corner algebra of A .

e.g. $A = M_2(K)$, $e = e_2$, then $e_2 A e_2 = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \cong K$

Lemma 2 For M a left A -module, we have $\text{Hom}_A(A, M) \cong M$ as A -modules and if $e \in A$ is idempotent, then $\text{Hom}_A(Ae, M) \cong eM$ as K -submodules. In particular $A \cong \text{End}_A(A)^{\text{op}}$ (if we used right modules, then we would not have to use the op here!) and $eAe \cong \text{End}_A(Ae)^{\text{op}}$. (4)

Pf: Send $\vartheta: A \rightarrow M$ to $\vartheta(1)$ and $m \in M$ to $a \mapsto am$, etc.

(detailed proof for right modules, see [ASS, Lemma 4.2])

Assom-Simson-Skowroński:
"Elements of the representation theory of associative algebras"

$$\psi: M \rightarrow \text{Hom}_A(A, M)$$

$$m \mapsto (a \mapsto am)$$

$$\Phi: \text{Hom}_A(A, M) \rightarrow M$$

$$\vartheta \mapsto \vartheta(1)$$

$$\Phi \circ \psi(m) = \Phi(a \mapsto am)$$

$$\psi \circ \Phi(\vartheta) = \psi(\vartheta(1)) = (a \mapsto a\vartheta(1))$$

Decompositions

We have already seen decomp. of $\text{Hom}_K(\bigoplus M_i, \bigoplus N_j)$ in Lemma 1.3.12
 $\leadsto$ can do the same for K -algebras A , and A -modules.

In part: For the algebra $\text{End}_A(M_1 \oplus \dots \oplus M_n)$, the projections onto the factors M_i give a complete family of orth. idempotents, and the Pierce decomposition is

$$\text{End}_A(M_1 \oplus \dots \oplus M_n) = \begin{bmatrix} \text{Hom}(M_1, M_1) & \dots & \text{Hom}(M_n, M_1) \\ \vdots & \ddots & \vdots \\ \text{Hom}(M_1, M_n) & \dots & \text{Hom}(M_n, M_n) \end{bmatrix}$$

$$\text{and } \text{End}_A(M^n) = M_n(\text{End}_A(M)).$$

$\leadsto$ if $M = X \oplus Y$, then $X = 0$ or $Y = 0$

Lemma 3 An A -module M is indecomposable $\Leftrightarrow \text{End}_A(M)$ has no nontrivial idempotents (other than 0 and 1).

ASS, cor 4.2

Pf: $\Leftarrow$: Assume $M = X \oplus Y$, then $\exists \pi_i: M \rightarrow X_i$ and $\exists \iota_i: X_i \hookrightarrow M$ s.t. $\iota_1 \pi_1 + \iota_2 \pi_2 = \text{id}_M$ but $\iota_i \pi_i$ are $\neq 0$ idempotents in $\text{End}_A M$.

$\Rightarrow$: Let M be (f.d.f.) indec., assume $\exists$ non-zero idemp $e_1, e_2 = 1 - e_1 \Rightarrow M \cong \text{Im}(e_1) \oplus \text{Im}(e_2)$
 $e_i: M \rightarrow M$
 $m \mapsto me_i$

§1.6: Extended example of construction of K -algebras: (5)

Path algebras Literature: [ASS], [ARS = Auslander-Reiten-Smalø], [Schiffler "Quiver Representations"]

Consider K -algebras, where K is a comm. ring.

Def 1.6.1: A quiver is a quadruple $Q = (Q_0, Q_1, h, t)$, where Q_0 is a set of vertices, Q_1 is a set of arrows, and $h, t: Q_1 \rightarrow Q_0$ are mappings specifying the head and tail vertices of each arrow.

$$\begin{matrix} & t(a) & a & h(a) \\ & \bullet & \longrightarrow & \bullet \end{matrix}$$

A path in Q of length $n > 0$ is a sequence $p = a_1 a_2 \dots a_n$ of arrows satisfying $t(a_i) = h(a_{i+1}) \quad \forall i \in \{1, \dots, n-1\}$

$$\begin{matrix} \bullet & \xleftarrow{a_1} & \bullet & \xleftarrow{a_2} & \bullet & \xleftarrow{a_3} & \dots & \bullet & \xleftarrow{a_n} & \bullet \end{matrix} \quad \text{or} \quad \begin{matrix} p = a_n \dots a_2 a_1 \\ \bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \dots \bullet \xrightarrow{a_n} \bullet \end{matrix} \quad t(a_{i+1}) = h(a_i)$$

Caution: a lot of literature uses the convention $p = a_1 a_2 \dots a_n$ for $\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \dots \bullet \xrightarrow{a_n} \bullet$ i.e. $h(a_i) = t(a_{i+1})$!

Our convention is for working with left modules and paths are written like composition of functions: $a_1 a_2$ means "first a_2 , then a_1 ".

The head and tail of p are $h(a_1)$ and $t(a_n)$.

$$p = \begin{matrix} & a_n & & a_2 & & a_1 \\ & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ t(p) & & & & & h(p) \end{matrix}$$

For each vertex $i \in Q_0$ there is a trivial path e_i of length 0 with $h(e_i) = t(e_i) = i$.

Now write KQ for the free K -module with basis all paths in Q . It has a multiplication of paths:

$$pq = \begin{cases} 0 & \text{if } t(p) \neq h(q) \\ pq & \text{the concatenation of } p \text{ and } q \\ & \text{otherwise} \end{cases}$$

explicitly $p = a_1 \dots a_n$ $q = b_1 \dots b_m$
 $\begin{matrix} a_1 & \dots & a_n & & b_1 & \dots & b_m \\ \bullet & \xleftarrow{a_1} & \bullet & \xleftarrow{a_2} & \dots & \bullet & \xleftarrow{b_1} & \dots & \bullet & \xleftarrow{b_m} & \bullet \end{matrix}$
 $\uparrow$
 $pq = a_1 \dots a_n b_1 \dots b_m$

In the following, we will always assume that Q is finite, i.e. $|Q_0| < \infty$.
 If not finite: then $\nexists$ identity $\leadsto KQ$ not unital
 as lemma 1.6.3

Rmk 1.6.2 :

A multiplication on KQ is given as follows:

assume that $(e_i)_{i \in I}$ is a o.s. basis of KQ . Take $a, a' \in KQ$:

$$a = \sum_{j \in I} \lambda_j e_j, \quad a' = \sum_{j' \in I} \lambda_{j'} e_{j'}, \quad \Rightarrow aa' = \sum_{j, j' \in I} \lambda_j \lambda_{j'} (e_j e_{j'})$$

mult. in K concat. if $t(e_j) = h(e_{j'})$

This mult. makes KQ a K -algebra, the path algebra of the quiver Q .

Lemma 1.6.3 : In a path algebra KQ , the identity element is given by the sum of all constant paths: $1_{KQ} = \sum_{i \in Q_0} e_i$.

Pf : Let $a \in KQ$, then $a = \sum_{j \in I} \lambda_j e_j$ with $\lambda_i \in K$. Then
 K -basis $(e_j)_{j \in I}$

$$\begin{aligned} \sum_{i \in Q_0} e_i \cdot a &= \sum_{i \in Q_0} \sum_{j \in I} e_i \lambda_j e_j = \sum_{i \in Q_0} \sum_{j: h(e_j)=i} \lambda_j (e_i e_j) \quad \text{where } e_i e_j = \begin{cases} 0 & \text{if } h(e_j) \neq i \\ e_j & \text{if } h(e_j) = i \end{cases} \\ &= \sum_{i \in Q_0} \sum_{j: h(e_j)=i} \lambda_j e_j = \sum_{j \in I} \lambda_j e_j = a. \end{aligned}$$

Similarly, show that $a \cdot \sum_{i \in Q_0} e_i = a$. □

Note : The e_i form a complete family of orthogonal idempotents.

Examples 1.6.4:

(1) Let Q be the quiver: $1 \mathcal{P}^\infty$. $|Q_0| = 1$ and paths in Q are $e, \alpha, \alpha^2, \alpha^3, \alpha^4, \dots$. Thus the algebra KQ has basis $\{\alpha^j, j \in \mathbb{N}\}$ and multiplication is given as $\alpha^s \alpha^t = \alpha^{s+t}$.
 $\Rightarrow KQ$ is commutative, in fact, $KQ \cong K[x]$.

(2) Let $Q: 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \rightarrow \dots \xrightarrow{\alpha_{n-1}} n$, the paths are:

(7)

Write as matrix: e_{ij} = path from j to i :

1: path j to 1 only
2: path j to 2 only
...

$$\begin{bmatrix} e_1 & 0 & 0 & \dots & 0 \\ \alpha_1 & e_2 & 0 & \dots & 0 \\ \alpha_2 \alpha_1 & \alpha_2 & e_3 & \dots & 0 \\ \alpha_3 \alpha_2 \alpha_1 & \alpha_3 \alpha_2 & \alpha_3 & e_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} \dots \alpha_1 & & & & & a_{n-1} e_n \end{bmatrix}$$

Here KQ is $\cong$ algebra of lower triangular matrices over K
 $M_n(K)$

(3) $Q: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$ gives KQ with basis: $e_1, e_2, a, b, ba, b^2, b^2a, b^3, \dots$

(4) $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$
Basis: e_{ij} : path j to i

$$\begin{bmatrix} e_1 & 0 & 0 & 0 \\ \alpha e_1 & e_2 & 0 & 0 \\ \alpha \beta e_1 & \beta e_2 & e_3 & 0 \\ \alpha \gamma e_1 & \gamma e_2 & 0 & e_4 \end{bmatrix}$$

Note that, as a K -vector space, KQ can be written

$$KQ = KQ_0 \oplus KQ_1 \oplus KQ_2 \oplus \dots \oplus KQ_\ell \oplus \dots$$

where $\forall \ell \geq 0$: KQ_ℓ is the subspace of KQ of all paths of length ℓ .
Clearly, $(KQ_n) \cdot (KQ_m) \subseteq KQ_{n+m} \quad \forall n, m \geq 0$ because the product of paths of length n and m is either 0 or of length $n+m$.

Sometimes we say that this defines a grading of KQ , or KQ is a graded K -algebra.

Lemma 1.6.4 Let Q be a quiver and KQ its path algebra. Then KQ is a finite dim'l K -algebra $\Leftrightarrow Q$ is finite and acyclic.

$\hookrightarrow$ doesn't contain oriented cycles, i.e., a path $p = a_1 \dots a_n$ s.t. $a_n = a_1$
 $a_1 \xrightarrow{\alpha_1} a_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_1$

Pf: If Q is infinite, then so is the basis

of KQ , so it is ∞ -dim'l over K (and KQ not unitary). If Q has a cycle $w = a_n \dots a_1$, then for each $t \geq 0$, we have a basis vector $w^t = (a_n \dots a_1)^t$, so that KQ is again ∞ -dim'l.

Conversely, if Q is finite and acyclic, it only contains finitely

many paths and so is finite dim'l over K . $\square$

(8)

Now we get to the modules of KQ :

KQ -modules are essentially the same as representations of Q :

K field $\Rightarrow V_i$'s are K -vector spaces

Def 1.6.5 A representation of Q is a tuple $V = (V_i, V_\alpha)$ consisting of a K -module V_i for each vertex i and a K -module map $V_\alpha: V_i \rightarrow V_j$ for each arrow $\alpha: i \rightarrow j$ in Q . (If there is no risk of confusion, write $\alpha: V_i \rightarrow V_j$ instead V_α).

If V is a fin. dim'l rep ($\dim(V_i) < \infty$), then its dimension vector is $\dim V = (\dim V_i) \in \mathbb{N}^{Q_0}$.

Examples:

(1) $Q: 1 \rightarrow 2$. Have representations $V: K \xrightarrow{1} K$, $V': K \xrightarrow{0} 0$, $V'': 0 \xrightarrow{0} K$, $V''': K^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} K^3$

with dim. vectors $\dim V = (1, 1)$, $\dim V' = (1, 0)$, $\dim V'' = (0, 1)$, $\dim V''' = (2, 3)$.

(2) $Q: 1 \rightarrow 2 \xrightarrow{3} 4$

lots of reps $V: 0 \xrightarrow{0} K \xrightarrow{0} 0$

or $V': K \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} K^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} K$

$V'': 0 \xrightarrow{0} K \xrightarrow{0} K$

Goal of quiver representation theory is to classify these representations.

i.e. as basic finite dim'l alg. $\Rightarrow KQ/I$

But these are "basically the same" as finite dim'l K -algebras, so classifying modules over these algebras is the same.

Correspondence. $KQ\text{-mods} \simeq \text{rep } Q$.

If V is a KQ -module, then $V = \bigoplus_{i \in Q_0} e_i V$ since e_i 's are complete set of idemp. can write any KQ -mod in this way.

We get a representation, also denoted V , with $V_i = e_i V$, and for any arrow $\alpha: i \rightarrow j$, the map $V_\alpha: V_i \rightarrow V_j$ is the map given by left multiplication by $\alpha \in e_j K Q e_i$.

path starting at i , ending at j .

Conversely, any representation V determines a KQ -module via

$V = \bigoplus_{i \in Q_0} V_i$ with the action of KQ given as:

- For $v = (v_i)_{i \in Q_0} \in V$ we have $e_i v = v_i \in V_i \subseteq V$. That is: mult. with the trivial path e_i corresponds to projection onto V_i and
- $\alpha_1 \alpha_2 \dots \alpha_n v = V_{\alpha_1} \circ V_{\alpha_2} \circ \dots \circ V_{\alpha_n} (v_{t(\alpha_n)}) \in V_{h(\alpha_1)} \subseteq V$.

Under this correspondence:

(1) KQ -module homom. $\theta: V \rightarrow W$ correspond to tuples (θ_i) consisting of a K -module map $\theta_i: V_i \rightarrow W_i$ for each vertex i satisfying $W_\alpha \theta_i = \theta_j V_\alpha$ for all arrows $\alpha: i \rightarrow j$.

$$\begin{array}{ccc} V_i & \xrightarrow{V_\alpha} & V_j \\ \theta_i \downarrow & \curvearrowright & \downarrow \theta_j \\ W_i & \xrightarrow{W_\alpha} & W_j \end{array}$$

$\nwarrow KQ = KQe_1 \oplus KQe_2$

Example: $KQ: 1 \rightarrow 2$, $V: K \xrightarrow{1} K$
 $V' \subseteq K \xrightarrow{0} 0$

KQ -mods: $K \oplus K$
 $K \oplus 0$

Have a map: $V \rightarrow V': K \xrightarrow{1} K$
 $\theta_1 = 1 \downarrow \quad \quad \downarrow \theta_2 = 0$
 $K \xrightarrow{0} 0$

We have $\theta_2 \circ \text{id}_K = 0 \circ \text{id}_K \checkmark$

But no map: $V' \rightarrow V: K \xrightarrow{0} K$
 $\theta_1 \downarrow \quad \quad \downarrow \theta_2$
 $K \xrightarrow{1} K$

because here $\theta_2 = 0$ injection $0 \rightarrow K$

$\text{id}_K \circ \theta_1 = \theta_1 \Rightarrow \theta_1 = 0$.

so $(\theta_1, \theta_2) = (0, 0)$, 0-homom.