WEEK 6

Recall from last week (we are in the process of proving Nakayama's lemma):

<u>Def 1.4.7</u> Let M be an R-module. A submodule $M' \subset M$ is called <u>superfluous</u> (in M) if for all submodules $N \subseteq M$ the equation $N + M' = M$ implies that $M = N$.

<u>Lemma 1.4.8</u> Let M be an R-module.

(1) TFAE: (a) M is finitely generated.

(b) $M/J(M)$ is f.g. and $J(M)$ is superfluous.

(2) $J(R)M \subset J(M)$.

(3) NAKAYAMA'S LEMMA: If M is finitely generated, then $J(R)M$ is superfluous.

<u>Pf</u>: (1) <u>(a) ⇒ (b)</u>: A factor module of a f.g. module is f.g. ⇒ $M/J(M)$ is f.g. For the second part, take $N \subsetneq M$. By Rmk 1.2.2. N is contained in a max. submodule M', i.e. $N \subseteq M' \subsetneq M$. By def., $J(M) \subseteq M'$, and so also $N + J(M) \subseteq M'$. Hence: $N + J(M) \subsetneq M$ for any $N \subsetneq M$. Thus, if $N + J(M) = M$, we must have that $N = M$.

already done!

<u>(b) ⇒ (a)</u> Let $n \geq 1$ and $x_1, \dots, x_n \in M$ such that $M/J(M) = \sum_{i=1}^{n} R(x_i + J(M))$ $\underbrace{}_{}$ M/J(M) cosets

Then $M = (\sum_{i=1}^{n} R x_i) + J(M)$. Since $J(M)$ is superfluous, it follows that $M = \sum_{i=1}^{n} R x_i$.

(2) For any $x \in M$, the map $R \xrightarrow{\cdot x} M : a \mapsto ax$, is an R-module homomorphism. By Lemma 1.4.3(1) $\substack{g: M_1 \to M_2 \\ g(J(M_1)) \subset J(M_2)}$, $J(R)x \subset J(M)$.

$\Rightarrow J(R)M \subset J(M)$.

(3) Suppose that M is finitely generated. Then by (1): $J(M)$ is

superfluous, and by (2): $J(R)\cdot M \subseteq J(M)$ is also superfluous. $\square$

Rmk: (3) can be reformulated: If M is f.g., then $J(R)M = M \Rightarrow M=0$. [Lj-W, Thm 3.17]

Next we characterize semi-simple rings with the help of the radical (Thm 1.4.10).

Thm 1.4.9 Let R be left artinian.

(1) $J(R)$ is the largest nilpotent ideal of R. $(\exists n \geq 1: I^n = 0)$

(2) $R/J(R)$ is a semi-simple ring.

(3) $J(R)M = J(M)$ for every R-module M.

(4) R is left noetherian.

Pf: (1) Every nilpotent ideal is nil, and hence contained in $J(R)$, by Lemma 1.4.6 (2). Thus it remains to show that $J(R)$ is nilpotent:
every nil ideal $\subseteq J(R)$

Clearly $R \supseteq J(R) \supseteq J(R)^2 \supseteq \dots$ is a descending chain of left ideals. By DCC $\exists n \in \mathbb{N}$, s.t. $J(R)^m = J(R)^n$ $\forall m \geq n$.

Set $I := J(R)^n$.

Claim: $I = 0$.

Assume on the contrary, that $I \neq 0$, and consider the set

$$\Omega = \{ J \subset R : J \text{ is left ideal with } I \cdot J \neq 0 \}.$$

Since $I^2 = J(R)^{2n} = J(R)^n = I \neq 0 \Rightarrow I \in \Omega$ (i.e. $\Omega \neq \emptyset$)

Thus there is a minimal element $J \in \Omega$. Further $\exists a \in J$ with $I \cdot a \neq 0 \Rightarrow I(Ra) \neq 0$. Since Ra is a left ideal with $Ra \subset J$ the minimality of J implies $Ra = J$.

In particular: J is f.g. and Nak. lemma $\Rightarrow J(R)J \subsetneq J$.

$\Rightarrow I \cdot J \subsetneq J$ and thus $I \cdot (IJ) = 0$ because of minimality of J.

But $I^2 = I \Rightarrow I \cdot (IJ) = I^2 \cdot J = IJ \neq 0$. ↯ $\Rightarrow$ Claim, i.e. $J(R)^n = 0$ nilpotent.

(2) By Cor 1.4.4 $R/J(R)$ is semi-simple R-mod. Since $J(R) \subset R$ is a two-sided ideal, $R/J(R)$ is a ring. Since every $R/J(R)$
R art. $\Rightarrow M/J(M)$ s.s.

module is also an R-module, $R/J(R)$ is a semi-simple $R/J(R)$-module, i.e., a semi-simple ring.

(3) $M/J(R)M$ is a $R/J(R)$-module. Since $R/J(R)$ is s.-s., $M/J(R)M$ is a s.-s. $R/J(R)$-module and hence a s.-s. R-module. Thus

M s.-s. $\Rightarrow J(M)=0$.

$$J(M/J(R)M)=0 \quad \text{by Rmk 1.4.2 (3).}$$

Now lemma 1.4.3 (2) $N\subset M$, with $N\subset J(M) \Rightarrow J(M/N)=J(M)/N$.

shows $0=J(M/J(R)M)=J(M)/J(R)M \Rightarrow J(M)=J(R)M$.

(4) For $i\in\mathbb{N}$, we set $M_i:=J(R)^i/J(R)^{i+1}$. ($M_0=R/J(R)$, $M_1=J(R)/J(R)^2, \ldots$)

Since M_i is a factor module of the submodule $J(R)^i$, it must be artinian. Since $J(R)\subset \mathrm{ann}_R(M_i)$

$\{x\in R : xJ(R)^i/J(R)^{i+1}=0\}$

$\hookrightarrow x\in J(R) : xJ(R)^i\subset J(R)^{i+1}$ ✓

$I\subset \mathrm{ann}\, M \Rightarrow M$ is R/I-module: add ✓ check mult (!)

$\forall x\in I: xm=0\ \forall m\in M$.

we obtain that M_i is an $R/J(R)$ module. Since $R/J(R)$ is s.-s., M_i is a s.-s. $R/J(R)$-module and hence a s.-s. R-module. Since M_i is artinian, M_i is a finite $\oplus$ of simple R-modules and hence M_i is noetherian R-module.

By (1): $\exists n\in\mathbb{N}$ s.t. $J(R)^{n+1}=0 \Rightarrow M_n=J(R)^n$ is a noeth. R-mod.

Use lemma 1.1.4 + induction $\Rightarrow J(R)^{n-1}, \ldots, J(R)^0=R$ are noeth. R-modules, thus R is a left-noeth. ring. $\square$

fact. mods of noeth mods are noeth.

Thm 1.4.10 TFAE:

(a) R is semi-simple.

(b) R is left (resp. right) artinian and has no nozero nilpotent ideals.

(c) R is left (resp. right) artinian and $J(R)=0$.

Pf: We prove left properties:

(a) ⇒ (b) By Thm 1.3.3. R is left artinian. By Thm 1.4.9, $J(R)$ is the largest nilpotent ideal of R. Since by Rmk 1.4.2. (3) $J(R)=0$, the assertion follows.

(b) ⇒ (c): $J(R)$ is nilpotent by Thm 1.4.9 (1) ⇒ $J(R)=0$.

(c) ⇒ (a): $R/\underbrace{J(R)}_{0}$ is semisimple by 1.4.9(2). $\square$

§1.5 Algebras

Def 1.5.1 Let K be a comm. ring (mostly: $K=k$ a field!) An (associative and unitary) K-algebra is a K-module A together with a multiplication $A\times A\to A : (a,b)\mapsto a\cdot b = ab$ s.t. the following hold:

(A1) $(A,+,\cdot)$ is a ring

(A2) For all $\lambda\in K$ and all $a,b\in A$, we have

$$\lambda(ab) = (\lambda a)b = a(\lambda b)$$

Alternatively: $A\times A\to A$ multipe. is K-bilinear

If $(A,+,\cdot)$ is a commutative (simple, semi-simple, ...) ring, then A is called a commutative (simple, semi-simple, ...) algebra. If K is a field and $\dim_K(A)<\infty$, then A is called a finite dimensional K-algebra. If $K\subset A$, then A is called central if $Z(A)=K$. → see remark 1.5.2(1) below!

Examples: (1) If K is a commutative ring, then $M_n(K)$ is a central (*) K-algebra, $K[X_1,\dots,X_n]$ is a comm. K-alg.

$G=\mu_2 \curvearrowright R$ via $a\mapsto -a$, $G\curvearrowright R[x]: f(x)\mapsto f(-x)$

(2) Consider $A = K[X]\oplus K[X]\sigma$ (generator) with multiplication

$(P(x)+Q(x)\sigma)(P'(x)+Q'(x)\sigma) =$
$[P(x)P'(x)+Q(x)Q'(x)] + [Q(x)P'(x)+Q'P]\sigma$

group ring: $K[X]\mu_2$

alternatively:
$(P(x)+Q(x)\sigma)(P'(x)+Q'(x)\sigma)$
$= P(x)P'(x) + Q(x)P'(-x)\sigma + P(x)Q'(x)\sigma + Q(x)Q'(-x) = (P(x)P'(x)+Q(x)Q'(-x)) + (Q(x)P'(-x)+P(x)Q'(x))\sigma$

is a $K[x]$-algebra (a skew group ring!): $K[x]*\mu_2$

→ Many more later!

Rmk 1.5.2: Let K be a commutative ring.

(1) If A is a K-algebra, then $\varepsilon: K\to A$, $\lambda\mapsto\lambda\cdot 1_A$ is a central ring homom. (i.e. $\varepsilon(K)\subset Z(A)$). If ε is injective (e.g. if K is a field), then we identify K and $K\cdot 1_A$.

Pf: For all $\lambda,\mu\in K$ and $a\in A$ we have

$$\varepsilon(\lambda\mu) = (\lambda\mu)\cdot 1_A = \lambda(\mu 1_A) = \lambda(1_A(\mu\cdot 1_A)) \overset{\text{alg. hom}}{=} (\lambda\cdot 1_A)(\mu\cdot 1_A) = \varepsilon(\lambda)\cdot\varepsilon(\mu).$$

and $\overset{\text{central}}{\varepsilon(\lambda)\cdot a} = (\lambda\cdot 1_A)a = \lambda(1_A a) = \lambda(a\cdot 1_A) = a\cdot\varepsilon(\lambda)$. □

So to clarify our notion of central algebra: A is central if $\varepsilon(K)=Z(A)$! i.e. $Z(A)=\{\lambda\cdot 1_A : \lambda\in K\}$

(2) Conversely, let A be a ring and $\varepsilon: K\to A$ be a central ring homom. Then A is a K-module and a K-algebra. In this case, also the structural homom. $\varepsilon: K\to A$ is called a K-algebra.

Moreover, every ring with $K\subset Z(A)$ is a K-algebra.

only check A2: $\lambda(ab) = (\lambda a)b \underset{K\subset Z(A)}{=} a(\lambda b)$

If A is a ring, then there is precisely one ring homom. $\varepsilon:\mathbb{Z}\to A$, namely: $m\mapsto m\cdot 1_A$ for $m\in\mathbb{Z}$), so any ring is a $\mathbb{Z}$-algebra.

(3) Let $\varepsilon_i: K\to A_i$ for $i=1,2$ be central ring homoms. A ring homom. $f: A_1\to A_2$ satisfying $f\circ\varepsilon_1=\varepsilon_2$ is called a <u>K-algebra homom</u>. (Equiv: f is a ring homom. and a K-module homom)

$$\begin{tikzcd} & K \arrow[dl, "\varepsilon_1"'] \arrow[dr, "\varepsilon_2"] & \\ A_1 \arrow[rr, "f"] \arrow[rr, phantom, shift right=3, "\circlearrowleft"] & & A_2 \end{tikzcd} \Longrightarrow$$

$f(a\cdot b) = f(a)\cdot f(b)$ etc. + "$f(\lambda a+\mu b)$" $= f(\varepsilon_1(\lambda)a + \varepsilon_1(\mu)b) = f(\varepsilon_1(\lambda))\cdot f(a) + f(\varepsilon_1(\mu))\cdot f(b)$

$f(\varepsilon_1(\lambda)a+\varepsilon_1(\mu)b) \underset{f\varepsilon_1=\varepsilon_2}{=} \varepsilon_2(\lambda)\cdot f(a) + \varepsilon_2(\mu) f(b)$

K-mod. hom $f(\lambda a+\mu b) = \underset{\lambda\cdot 1_{A_2}=\varepsilon_2(\lambda)}{\lambda} f(a) + \mu f(b)$

(4) If $k \subset A_1$ and $k \subset A_2$, then a ring homom. $f: A_1 \to A_2$ is a k-algebra homom $\iff f|_k = \mathrm{id}_k$.

Pf: $\Rightarrow$: If f is a k-alg. hom. and $a \in k$, then
$$f(a) = f(\underbrace{a \cdot 1_{A_1}}_{\varepsilon_1(a)}) = a \cdot f(1_A) = a \cdot 1_{A_2} = a.$$

$\Leftarrow$: If $f|_k = \mathrm{id}_k$, $a \in k$ and $x \in A_1$ then $f(ax) = f(a) \cdot f(x) = a \cdot f(x)$. k-mod hom ✓ □

(5) Let A be a k-algebra and $I \subset A$ be a left ideal (resp. right). Then $I \subset A$ is a k-submodule.

Pf: Let $I \subset A$ be a left ideal. We have to check that $\forall \lambda \in k$ and $\forall x \in I$, we have $\lambda x \in I$.
Indeed: $\lambda x = \lambda(1_A x) = (\lambda 1_A) x \subset AI \subset I$
If I is a right ideal, then $\lambda x = \lambda(x \cdot 1_A) = x(\lambda \cdot 1_A) \subset IA \subset I$ □

Lemma 1.5.3 Let k be a comm. noetherian (resp. artinian) ring, and A be a k-algebra, which is f.g. as a k-module (this is sometimes called module finite over k). Then A is left and right noetherian (resp. artinian). In part., finite dim'l k-algebras over a field k are artinian rings.

Pf: We show that A is left noetherian. Since left ideals are k-submodules and A is noeth. over k (f.g. over k noeth ✓) $\Rightarrow$ left ideals satisfy the ACC, hence A is left noetherian. □

Prop 1.5.4 Let k be a field and A a finite dim'l k-algebra.
(1) A is left artinian, right artinian, left noetherian and right noetherian.
(2) If $a, b \in A$ with $ab = 1$, then $ba = 1$.
(3) Every left zero-divisor is a right zero divisor and conver=

sely. Every element of A is either invertible or a zero-divisor. If A has no nonzero zero-divisors, then A is a division algebra.

(4) If A is a division algebra, then its center is a field.

Pf: (1) See Lemma 1.5.2.

(2) This follows from (1) and ex. 1.1.9 (6) If R is left noeth. and $a, b \in R$ with $ab = 1 \Rightarrow ba = 1$

(3) Suppose that $a \in A$ is not a left zero-divisor. Then the map $f: A \xrightarrow{a\cdot} A : x \mapsto a \cdot x$ is K-linear and injective, $\Rightarrow$ surjective.

A = f. dim. v.s. over field $K \rightsquigarrow f: K^n \to K^n$ inj $\Rightarrow$ surj.
$A \cong K^n$
because let $v_1, \dots, v_n$ be basis of A as K-v.s. $\rightsquigarrow f$ given by $\underbrace{f(v_1), \dots, f(v_n)}_{\text{basis of } A}$

$\Rightarrow$ any $a \in A$ can be written $a = \sum_{i=1}^{n} \lambda_i f(v_i) = \underbrace{\sum_{i=1}^{n} f(\lambda_i v_i)}_{K\text{-lin}} \in \operatorname{im}(f)$.

Thus $\exists b \in A$ s.t. $ab = 1$ and by (2) also $ba = 1$. Then a is not right z.d.

Similarly, if $a \in A$ is not a right zero divisor, then a is invertible. Since $zd(A) \cap A^* = \emptyset$ the remaining assertion follows.

(4) Every subalg. of a fin. dim'l algebra is a division alg. since it is fin dim'l again and so by (3) a division algebra. Hence its center is a fin'dim'l comm. division algebra (i.e. a field).

Example 1.5.5

(1) Every finite field extension L/K is a fin. dim'l K-algebra: e.g. $\mathbb{Q}[\sqrt{5}] = \mathbb{Q} \oplus \mathbb{Q}\sqrt{5}$ is 2-dim'l $\mathbb{Q}$-algebra
$\mathbb{R}[i] = \mathbb{R} \oplus \mathbb{R}i \cong \mathbb{C}$ is a 2-dim'l $\mathbb{R}$-algebra
1-dim'l $\mathbb{C}$-algebra

(2) Let K be a field and A be a 2-dim'l K-algebra which is not a field. Then A is commutative and isomorphic either to $K \times K$

or $k[x]/(x^2)$. (see e.g. [Drozd-Kirichenko

(3) Every finite division ring is commutative. nice proof e.g. in [Aigner-Ziegler: Proofs from the BOOK]

(4) Let A be a finite dimensional division algebra over $\mathbb{R}$. Then A is isomorphic either to $\mathbb{R}$, $\mathbb{C}$ or to Hamilton's quaternions see e.g. [Jantzen-Schwermer, Theorem 1.11].

Rmk 1.5.6: Let k be a field.

(1) If D is a division ring, then the center of $M_n(D)$ is isom. to the center of D (cf. ex. 1.3.5). By Prop 1.5.4, the center of D is a field. equiv: R simple left art $\Leftrightarrow R \cong M_n(D)$ for D div. ring

Thus, by Cor 1.3.16 the center of a simple artinian k-algebra is a field.

If A is a simple artinian k-algebra with center $Z(A)$, then $k \subset Z(A)$ ← field and thus A is a simple artinian $Z(A)$-algebra. Thus the study of simple artinian algebras reduces to the study of central simple artinian algebras.

(2) Let A be a fin. dim'l k-algebra. Then A is simple $\Leftrightarrow A \cong M_n(D)$ where $n \in \mathbb{N}_{>0}$ and D is a division algebra (clearly: $M_n(D)$ is simple; conversely, if A is simple, then it is $\cong$ to matrix ring over a division algebra by Cor 1.3.16 because A is left artinian by Lemma 1.5.3) Note that $n^2 \dim_k D = \dim_k A < \infty$.

(3) Let D be a finite dim'l division algebra over k. Then every commutative subalgebra of D is a field. If k is algebraically closed, then $D = k$.

Pf: Let $E \subset D$ be a fin. dim'l subalgebra, then by same arg. as for 1.3.4 (4), E is a division algebra.

If E is comm $\Rightarrow$ E is a field with $k \subset E$.

If k is alg. closed, then $k = E$.

Now suppose that k is alg. closed. If $a \in D$, then $E = k[a]$ is a comm. subalgebra of D. By the above, $E = k$, thus $a \in k$.
$\Rightarrow D = k$. □

(4) Let k be alg. closed and A a fin. dim'l simple k-algebra. Then $A \cong M_n(k)$. In part., $Z(A) \cong k$ and $\dim_k(A) = n^2 \cdot \dim_k k$
$= n^2$.

Follows from (2) $A \cong M_n(D)$; (3) $D = k$.

More generally: if A is a finite dim'l central simple k-algebra, then $\dim_k A$ is a square. (see e.g. [Jantzen Schwermer p269])

<u>Sketch</u>: For an extension field L/k, we get
$$\dim_k A = \dim_L(A \otimes_k L) = \dim_L(M_n(L)) = n^2$$

(5) If $\mathrm{char}(k) \neq 2$ and A is a central simple k-algebra with $\dim_k A = 4$, then A is isomorphic to a quaternion algebra
e.g. [Maclachlan-Reid "Arithmetic of Hyperbolic 3-Manifolds"].
Thm 2.1.8