WEEK 5

Last week: Discussed how to make a <u>right</u> R-module M into a <u>left</u> R^{op}-module:

Define a map: $R^{op} \times M \to M$

$$(a, m) \mapsto a \cdot^{op} m = m \cdot a \quad \forall a \in R, \forall m \in M.$$

<u>This</u> makes M a left R^{op}-module: $\forall a, b \in R, \forall m \in M$

This week: Artin-Wedderburn structure theorem. But first:

<u>Theorem 1.3.15</u> Let R be simple and artinian, M a simple (left) R-module, $D = \mathrm{End}_R(M)$ and $n = \underbrace{\dim_D(M)}$.

this is well-def → every module over a div. ring is free ⇝ dim = rk of this free mod

Then: $R \cong \mathrm{End}_D M \cong M_n(D^{op})$ as rings.

<u>Proof</u>: Define $\Phi : R \to \mathrm{End}_D M$

$$a \mapsto l_a : M \to M$$
$$x \mapsto a \cdot x$$

this is a ring hom (!)

$\Phi(a+b)(x) = (a+b)(x) = ax + bx = \Phi(a)(x) + \Phi(b)(x)$.

$\Phi(ab)(x) = (ab)x = abx$

$\Phi(a)(\Phi(b)(x)) = \Phi(a)(bx) = abx$

$\Phi(1)(x) = x = \mathrm{Id}_{\mathrm{End}_D M}$ ✓

let $a \in \ker\Phi$, $\forall b \in R$ then $\Phi(ba) = bax = b0 = 0 \ \forall x \in M$; $\Phi(ab) = a \cdot (bx) = a\tilde{x} = 0$ ✓

Since $\ker \Phi = \{a \in R : ax = 0 \ \forall x \in M\}$ is a <u>two-sided ideal</u> in R and $1 \notin \ker\Phi$, it follows (from R simple), that $\ker \Phi = 0$.

$\Phi(1)(x) = \mathrm{Id}_{\mathrm{End}_D M} \neq 0$ since $M \neq 0$ ← M simple

$\Rightarrow \Phi$ is monomorphism.

In order to show surjectivity, we want to use Jacobson density thm, but for this we need finitely <u>many</u> elements, therefore

Claim: $\dim_D(M) < \infty$.

Pf of Claim: Let $s \in \mathbb{N}$, $\{m_1, \dots, m_s\} \subset M$ be D-linearly idependent and for $i \in \{1, \dots, s\}$ define

$$A_i = \{a \in R : a m_j = 0 \quad \forall j \in \{1, \dots, i\}\} \subset R$$

Clearly, $A_s \subset R$ is a left ideal. Since $m_1, \dots, m_s$ are D-linearly independent, there is, for all $y_1, \dots, y_s \in M$, some $f \in \mathrm{End}_D M$ with $f(m_i) = y_i \;\; \forall i \in \{1, \dots, s\}$ [f is defined by values of a basis $\rightsquigarrow$ part of basis also gives map]

in part: $f(m_i) = 0 \;\; \forall i = 1, \dots, s$

By Thm 1.3.14, $\exists a \in R$ with $f(m_i) = a m_i \overset{=0}{} \;\; \forall i \in \{1, \dots, s\}$.

$$\Rightarrow A_1 \supsetneq A_2 \supsetneq \dots \supsetneq A_s.$$

$A_1 = \{a \in R : a m_1 = 0\}$ Choose $y_1 = 0$, others don't care

$A_2 = \{a \in R : a m_1 = a m_2 = 0\} \rightarrow y_1 = 0, y_2 = 0$

If $a \in A_2 \Rightarrow a m_1 = 0 \Rightarrow a \in M_2$

If we choose $y_1 = 0, y_2 \neq 0 \rightsquigarrow$ get a with $a m_1 = 0$, $a m_2 \neq 0 \Rightarrow a \in A_1 \setminus A_2$ etc.

If there was an infinite set of D-linearly independent elements, then we would have an infinite descending chain of left ideals. $\lightning$ to R artinian. $\square$ Claim

Now let $\{m_1, \dots, m_n\}$ be a D-basis of M. Again, by Thm 1.3.14 for every $f \in \mathrm{End}_D(M)$, there is an $a \in R$ with $f(m_i) = a m_i \;\; \forall i = 1, \dots, n$

$\Rightarrow f(m) = am = \ell_a(m) \;\; \forall m \in M$. Hence Φ is surjective.

$\Rightarrow \Phi$ is a ring isomorphism.

Since M is a free D-module of rk n. $M \cong D^n$

Lemma 1.3.12 (4)

$\Rightarrow \mathrm{End}_D M \cong \mathrm{End}_D(D^n) \cong M_n(\mathrm{End}_D(D))$ as rings

$\mathrm{End}_R(M^n) = M_n(\mathrm{End}_R M)$

So it just remains to show that: $\mathrm{End}_D D \cong D^{op}$ (as rings).

Consider the map $\psi: D^{op} \to \mathrm{End}_D D$
$$a \mapsto f_a: D \to D : x \mapsto x \cdot a$$

ψ is inj. $[\psi(a)(x) = 0 \ \forall x \in D \Rightarrow x \cdot a = 0 \ \forall x \in D \Leftrightarrow a = 0]$

For any $g \in \mathrm{End}_D D : (x \mapsto g(x))$, then $\forall x \in D: g(x) = g(x \cdot 1) = x \cdot g(1)$

$\Rightarrow g(x) = \underbrace{\psi(g(1))(x)}_{x \cdot g(1)} \Rightarrow \psi$ is surjective.

ψ is a ring hom: additive ✓

$$\psi(1_D) = (x \mapsto x \cdot 1_D = x) = \mathrm{Id}_{\mathrm{End}_D M} \ ✓$$

$$\psi(a \cdot^{op} b)(x) = \psi(b \cdot a)(x) = x \cdot (b \cdot a) = x \cdot b \cdot a$$

$$\psi(a)(\psi(b)(x)) = \psi(a)(x \cdot b) = (x \cdot b) \cdot a = x \cdot b \cdot a. \ ✓$$

$\Rightarrow D^{op} \cong \mathrm{End}_D(D)$. $\square$

<u>Cor 1.3.16</u> For a simple ring R, TFAE:

Thm 1.3.11
(a) R is left artinian.
(b) R has a minimal left ideal.
(c) R is semi-simple.
(d) R is isomorphic to a matrix algebra over a division ring.

Pf: This follows from Thms 1.3.11, 1.3.15, and Ex. 1.3.5.

<u>Theorem</u> 1.3.17 (Artin-Wedderburn) or: Wedderburn-Artin or "WHAM": 1893 Molien: Thm for $\mathbb{C}$. fin. dim. algs
1907: Wedderburn: f. dim. algs over arb. fields
1927: E. Artin: ACC+DCC
1939: Hopkins: ACC follows from DCC

TFAE:

(a) R is semi-simple.

(b) $R \cong M_{n_1}(D_1) \times \dots \times M_{n_s}(D_s)$ where $s, n_1, \dots, n_s \in \mathbb{N}$ and $D_1, \dots, D_s$ are division rings.

Moreover, if R satisfies these conditions, then, $s, n_1, \dots, n_s$, and $D_1, \dots, D_s$ are uniquely determined.

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Pf: (a) $\Rightarrow$ (b) By Thm 1.3.10 R is a direct sum of simple rings, and each of them is semi-simple. ($R = \bigoplus_{i=1}^{r} R_i \rightsquigarrow$ each $R_i = \bigoplus_{j=1}^{n_i} L_{ij}$ the L_{ij} are also R_i-mod. simple)

The statement follows from Cor 1.3.16.

(b) $\Rightarrow$ (a) By our example 1.3.5, matrix rings over division rings are semi-simple (and simple!) and thus their product is semi-simple.

Proof of uniqueness: Suppose that

$$M_{n_1}(D_1) \oplus \dots \oplus M_{n_s}(D_s) \cong M_{m_1}(D_1') \oplus \dots \oplus M_{m_r}(D_r') \text{ as rings,}$$

where $r, s, n_1, \dots, n_s, m_1, \dots, m_s \in \mathbb{N}_{\geq 1}$ and D_i, D_i' are division rings. Since this is an isom. of semi-simple rings, their 2-sided ideals coincide $\overset{\text{Thm 1.3.10}}{\Longrightarrow}$ $r = s$ and (after possible renumbering) $M_{n_i}(D_i) \cong M_{m_i}(D_i')$ $\forall i \in \{1, \dots, s\}$, so we only have to verify the following:

Claim: Let D, D' division rings, $m, n \geq 1$, such that $M_n(D) \underset{\substack{\downarrow \\ \text{assume this} \\ \text{is called } \varphi}}{\cong} M_m(D')$. Then $m = n$ and $D \cong D'$.

Pf of claim: Set $R := M_n(D)$ and $R' = M_m(D')$. Then

$$L = \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D & 0 & \cdots & 0 \end{bmatrix}}_{n} \Big\} n = Re \text{ with } e = \underbrace{\begin{bmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{bmatrix}}_{\text{idempotent}} \in R.$$

L is a min. left ideal of R. Similarly,

$$L' = \begin{bmatrix} D' & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D' & 0 & \cdots & 0 \end{bmatrix} = R'f \text{ is min. left ideal in } R',$$

The given ring isomorphism $\varphi: R \to R'$, $\underset{\text{idemp. of } R}{Re} \mapsto \underset{\text{idemp. of } R'}{Re'}$ $\rightarrow$ $R \xrightarrow{\varphi} R' \xrightarrow{\cong} R'$ s.t. $e' \mapsto f$

induces (via base change in R') an isom $\psi: R \to R'$ mapping $e \mapsto f$ and L to L' and hence $D \cong eRe \mapsto fR'f \cong D'$

Finally: $\dim_D(R) = n^2$, $\dim_{D'}(R') = m^2$ $\Big\}$ $n^2 = m^2 \Rightarrow n = m$. □

Remark 1.3.18

(1) By Thm 1.3.17 we also obtain the following symmetric statements: For a ring R TFAE:

(a) R is semi-simple (i.e. R is s.s. left-R-module, written ${}_RR$)

(b) R is a finite direct product of simple left artinian rings.

(c) $R \cong M_{n_1}(D_1) \oplus \dots \oplus M_{n_s}(D_s)$, with parameters as in 1.3.17

(d) R_R is a semi-simple right module.

(e) R is a finite direct product of simple right artinian rings.

Similar statements for simple rings

(2) A commutative semi-simple ring is isomorphic to a finite direct product of fields.

(3) A semi-simple ring having no nonzero zero-divisors is a division ring.

§1.4 Jacobson Radical

Def 1.4.1. Let M be an R-module. The Jacobson radical $J(M)$ of M is the intersection of all maximal submodules of M. proper! ($N \subsetneq M$)
(Convention: If M doesn't have any max. submodules, then $J(M)=M$).

Examples 1.4.2

(1) From comm. algebra: $J(R) = \bigcap_{m \in \mathrm{Specmax}(R)} m$ (R comm.)

$\rightsquigarrow$ If R is local, i.e. (R, m) only has one max. ideal then $\mathcal{J}(R) = m$

(2) Set $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \subseteq M_2(k)$ upper Adler matrices

left ideals $\underbrace{\begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}}_{\text{max}}$, $\underbrace{\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}}_{\text{not max}}$, $\underbrace{\begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}}_{\text{max}}$

$$\begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix} \cap \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} = \mathcal{J}(R).$$

(3) If M is semi-simple, then $\mathcal{J}(M) = 0$.

Pf: If M is simple, then 0 is the only max. submodule $\Rightarrow \mathcal{J}(M) = 0$.
If $M = \bigoplus_{i \in I} M_i$, all M_i simple, $|I| \geq 2$, then $\forall i \in I$, $N_i = \bigoplus_{j \in I \setminus \{i\}} M_j$ is a maximal submodule of M. Clearly $\bigcap_{i \in I} N_i = 0$. $\square$

(4) $\mathcal{J}(\mathbb{Z}) = \bigcap_{p \text{ prime}} p\mathbb{Z} = 0$

(5) $\mathcal{J}(M) = \bigcap \ker \varphi$, where the intersection runs over all $\varphi: M \to E$ with E simple.

Pf: If $0 \neq \varphi: M \to E$, where E is simple, then φ must be surjective [$\mathrm{im}(\varphi)$ is submodule of E] $\underset{\text{Hom. Thm}}{\Longrightarrow}$ $E \cong M/\ker\varphi$.

$\Rightarrow \ker\varphi$ is maximal [see Rmk 1.2.2 (2)(c)]

Conversely, every max. submodule $N \subset M$ is the kernel of the projection $\pi: M \to \overbrace{M/N}^{\text{simple by Rmk 1.2.2}}$.

Lemma 1.4.3 (1) If $\varphi: M_1 \to M_2$, then $\varphi(\mathcal{J}(M_1)) \subseteq \mathcal{J}(M_2)$.

(2) If $N < M$ is a submodule with $N \subset \mathcal{J}(M)$, then $\mathcal{J}(M/N) = \mathcal{J}(M)/N$.

(3) $\mathcal{J}(M/\mathcal{J}(M)) = 0$.

(4) If $(M_i)_{i \in I}$ is a family of R-modules, then $\mathcal{J}(\bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} \mathcal{J}(M_i)$.

Pf: (1) For every homom. $\psi: M_2 \to E$, where E is simple, we get a homom. $\psi \circ \varphi: M_1 \to E \underset{1.4.2.(5)}{\Longrightarrow} \mathcal{J}(M_1) \subset \ker(\psi \circ \varphi)$

$M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} E$

$\Rightarrow \varphi(\mathcal{J}(M_1)) \subset \ker(\psi)$.

Set $m_1 \in \mathcal{J}(M_1) \Rightarrow \psi(\varphi(m_1)) = 0$
$\psi(\varphi(\mathcal{J}(M_1))) = 0 \Rightarrow \varphi(\mathcal{J}(M_1)) \subset \ker\psi$.

Since $J(M_2) = \bigcap_{\substack{f: M_2 \to S \\ S \text{ simple}}} \ker(f) \Rightarrow \varphi(J(M_2)) \subseteq \ker(\psi)$.

(2) The maximal submodules of M/N are of the form M'/N with $N \subseteq M'$ and $M' \subset M$ maximal. Since $N \subseteq J(M)$, for any such max M' we have $N \subset M'$. Thus $J(M/N) = \bigcap_{\substack{M' \subset M \\ \text{max.}}} M'/N$. By def. $J(M)/N := (\bigcap_{\substack{M' \subseteq M \\ \max}} M')/N$

$= \bigcap_{M' \subseteq M \max} M'/N$.

(3) Set $N = J(M)$ in (2): $J(M/J(M)) = J(M)/J(M) = 0$. □

(4) We set $M := \bigoplus_{i \in I} M_i$. Consider $\varepsilon_i : M_i \hookrightarrow M$ inclusion. Then by (1),
$\underbrace{\varepsilon_i(J(M_i))}_{J(M_i)} \subseteq J(M) \; \forall i \in I \Rightarrow \bigoplus_{i \in I} J(M_i) \subseteq J(M)$.

For the other inclusion, consider $p_i : M \to M_i$ projection.
Then (1) says that $p_i(J(M)) \subseteq J(M_i) \subseteq \bigoplus_{i \in I} J(M_i)$ for any $i \in I$.
and hence $J(M) \subseteq \bigoplus_{i \in I} J(M_i)$. □

Cor 1.4.4. If M is an artinian R-module, then $M/J(M)$ is semi-simple.
Pf: Since M is artinian and intersections of max modules forms a descending chain ($M_1 \supseteq M_1 \cap M_2 \supseteq M_1 \cap M_2 \cap M_3 \supseteq \dots$ for any submods $M_i \subseteq M, i \in I$), the chain of inters. of max. submods must become stationary, so it is enough to consider finitely many:
$J(M) = \bigcap_{i=1}^{n} M_i$ for some $M_i \subseteq M$ max.. Thus have a. seq:

$$0 \to J(M) \to M \to \bigoplus_{i=1}^{n} \underbrace{M/M_i}_{M_i \text{ max in } M \Rightarrow M/M_i \text{ is simple}}$$

From this get injective map $\varphi : M/J(M) \to \bigoplus_{i=1}^{n} M/M_i$
$m + J(M) \mapsto (m + M_i)$

$\Rightarrow \varphi(M/J(M))$ ($\cong M/J(M)$ since φ injective!) is isom. to a submodule of a s.-s. module and thus itself semi-simple. □

Def 1.4.5

(1) An element $a \in R$ is called <u>nilpotent</u> if $\exists n \geq 1$ s.t. $a^n = 0$.

(2) A subset $I \subset R$ is called <u>nil</u> if every $a \in I$ is nilpotent.

(3) A left (resp. right, resp. two-sided) ideal I is called <u>nilpotent</u> if there is an $n \geq 1$ such that $I^n = 0$.

<u>Examples</u>: • By def. every nilpotent ideal is nil.

• In $R = \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{bmatrix}$ = ring of upper triangular matrices, any matrix $\begin{bmatrix} 0 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 0 \end{bmatrix}$ is nilpotent. Hence $I = \begin{bmatrix} 0 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 0 \end{bmatrix}$ is nilpotent 2-sided ideal

<u>Lemma 1.4.6</u> (1) $J(R) = \bigcap_{E \text{ simple}} \overbrace{\mathrm{ann}_R(E)}^{\{r \in R \,:\, r \cdot e = 0 \ \forall e \in E\}} \subset R$ is a 2-sided ideal

(2) If $I \subseteq R$ is a nil left ideal, then $I \subset J(R)$.

<u>Pf</u> (1) Any annihilator $\mathrm{ann}_R(M)$ of a left R-module is a 2-sided ideal: $\mathrm{ann}_R(M) = \{x \in R : x \cdot m = 0 \ \forall m \in M\}$

Let $r \in R$, $x \in \mathrm{ann}_R(M)$: $(r \cdot x) \cdot m = r(xm) = r \cdot 0 = 0 \Rightarrow rx \in \mathrm{ann}_R(M)$

$(x \cdot r) m = x \cdot \underbrace{(rm)}_{\in M} = 0 \Rightarrow xr \in \mathrm{ann}_R(M)$.

Thus enough to show $J(R) = \bigcap_{E \text{ simple}} \mathrm{ann}_R(E)$.

isom. thm: $Rx \cong R/\mathrm{ann}_R x$. $\quad 0 \to \mathrm{ann}_R x \to R \xrightarrow{\cdot x} R \to 0$

If E is simple, then write $E = Rx$ for any $x \neq 0 \in E \Rightarrow E \cong R/\mathrm{ann}_R(x)$.

$\Rightarrow \mathrm{ann}_R(x)$ is a maximal left ideal of R.

Conversely, if $I \subseteq R$ is a maximal left ideal, then R/I is a simple R-module and $I = \mathrm{ann}_R(1 + I)$. $\quad x \in \mathrm{ann}_R(1+I) \iff x \cdot (1+I) = 0 + I \iff x + I = 0 + I \iff x \in I$.

Thus we have $J(R) = \bigcap_{E \text{ simple}} \left(\bigcap_{x \in E} \mathrm{ann}_R(x) \right) = \bigcap_{E \text{ simple}} \mathrm{ann}_R(E)$.

(2) Let $a \in I$. By (1) it suffices to show that $aE = 0$ $\forall E$ simple.
Let E be a simple R-module and assume on the contrary that $\exists x \neq 0 \in E$ with $ax \neq 0$. Then $R(ax) = E$, and thus $\exists b \in R$

s.t. $b(ax)=x$. Then $(ba)^2x = ba(bax) = bax = x \rightsquigarrow (ba)^n x = x\ \forall n$.
But $a \in I$ implies $ba \in I$ and ba nilpotent, i.e. $\exists m \geq 1$ s.t. $(ba)^m = 0$ $\lightning$. $\square$

Now we will approach the noncomm. Nakayama lemma, we need some notation first:

<u>Def 1.4.7</u> Let M be an R-module. A submodule $M' \subseteq M$ is called <u>superfluous</u> (in M) if for all submodules $N \subseteq M$ the equation $N + M' = M$ implies that $M = N$.

<u>Lemma 1.4.8</u> Let M be an R-module.

(1) TFAE: (a) M is finitely generated.

(b) $M/J(M)$ is f.g. and $J(M)$ is superfluous.

(2) $J(R)M \subset J(M)$.

(3) NAKAYAMA'S LEMMA: If M is finitely generated, then $J(R)M$ is superfluous.

<u>Pf</u>: (1) <u>(a) ⇒ (b)</u>: A factor module of a f.g. module is f.g. $\Rightarrow M/J(M)$ is f.g. For the second part, take $N \subsetneq M$. By Rmk 1.2.2. N is contained in a max. submodule M', i.e. $N \subseteq M' \subsetneq M$.
By def., $J(M) \subseteq M'$, and so also $N + J(M) \subseteq M'$. Hence:
$N + J(M) \subsetneq M$ for any $N \subsetneq M$. Thus, if $N + J(M) = M$, we must have that $N = M$.

<u>(b) ⇒ (a)</u> Let $n \geq 1$ and $x_1, \dots, x_n \in M$ such that $M/J(M) = \sum_{i=1}^{n} R(x_i + J(M))$ ($\underline{M/J(M)\text{ cosets}}$)
Then $M = (\sum_{i=1}^{n} R x_i) + J(M)$. Since $J(M)$ is superfluous, it follows that $M = \sum_{i=1}^{n} R x_i$.