

Week 2

Recall: last week noetherian, artinian modules + free modules ①

This week we will define simple + semi-simple modules, in order to study the structure of artinian + (semi-)simple rings

Lemma 1.1.6 (Hilbert's basis theorem) If R is left noetherian, then $R[X]$ is left noetherian.

Pf: Adept comm. Pf, see [G-W] Thm 1.9.

Lemma 1.1.7. Let R be a division ring and M an R -module.

TFAC: (a) M is noetherian.

(b) M is artinian.

(c) $\dim_R(M) < \infty$.

well-def by Lemma 1.1.2: every R -module has a basis

Pf: Adept comm. proof (uses f.c. modules and composition series $\leadsto$ see [Atiyah-MacDonald, Prop 6.10]).

Remark 1.1.8 (1) Subrings of noetherian (artinian) rings need not be noetherian (artinian). Simple ex: $R[x, y]$ is noeth., but $A = R[xy, xy^2, xy^3, \dots] \subset R[x, y]$ is not noeth: $0 \subset (xy) \subset (xy, xy^2) \subset \dots$

(2) Let $R \subset S$ be a ring extension. If S is comm.

+ noetherian and module finite (i.e. a f.g. R -module), then R is noetherian by the thm of Eakin-Nagata.

Example 1.1.9

(1) Consider $\Lambda = M_2(R)$, R any ring.

Then $I_1 := \Lambda e_1 = \begin{bmatrix} R & R \\ R & R \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & 0 \\ R & 0 \end{bmatrix} = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, a, b \in R \right\}$ is a left ideal.

*↑
idempotent:
see later*

Similar: $I_2 := \Lambda e_2 = \begin{bmatrix} R & R \\ R & R \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & R \\ 0 & R \end{bmatrix}$ is a left ideal (but not right!)

$\leadsto \Lambda = I_1 \oplus I_2$

For the other side: $e_1 \Lambda = \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix}$ and $e_2 \Lambda = \begin{bmatrix} 0 & 0 \\ R & R \end{bmatrix}$ are right ideals.

(2) Let $n \in \mathbb{N}$. If $I \subseteq R$ is a left ideal, then $M_n(I) \subseteq M_n(R)$ is a left ideal. [Pf: exercise: $n=2$ $\begin{bmatrix} I & I \\ I & I \end{bmatrix} \subseteq \begin{bmatrix} R & R \\ R & R \end{bmatrix}$ check axioms.]

(3) Let $n \in \mathbb{N}$. Then R is left noetherian $\Leftrightarrow M_n(R)$ is left noetherian.

Pf (Sketch): $\Rightarrow$: as R -module: $M_n(R) \cong R^{n^2}$ $\sim$ cor 1.1.5 (3)

$\Leftarrow$: If $(I_j)_{j \in \mathbb{J}}$ is an ascending chain of ideals in R , then $M_n(I_j)_{j \in \mathbb{J}}$ is an ascending chain of ideals in $M_n(R)$ $\Rightarrow M_n(R)$ has ACC $\Rightarrow R$ has ACC. $\square$

Note: R and $M_n(R)$ have "the same module category" $\sim$ they are Morita equivalent.

(4) The ring $\begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix} \subset M_2(\mathbb{Q})$ is right noetherian, but not left noetherian ($\mathbb{Z} \otimes \mathbb{Q}$ is not noetherian!) asc. chain of submodules e.g. $\langle \frac{1}{2} \rangle \subsetneq \langle \frac{1}{4} \rangle \subsetneq \dots$

The ring $\begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix} \subset M_2(\mathbb{R})$ is right artinian, but not left artinian.

(5) Let M be left noetherian and $f: M \rightarrow M$ surjective. Then f is an isomorphism.

Pf: Consider $f^n(x) = \underbrace{f \circ \dots \circ f}_n(x)$: For $n \in \mathbb{N}$, set show f inj!

$$M_n = \ker(f^n) = \{x \in M : f^n(x) = 0\}.$$

Then if $x \in M_i \Rightarrow f^i(x) = 0 \Rightarrow \forall k \geq 0: f^{i+k}(x) = f^k(f^i(x)) = f^k(0) = 0$.

$\Rightarrow M_1 \subseteq M_2 \subseteq \dots$ is AC $\Rightarrow \exists N \in \mathbb{N}$ s.t. $M_N = M_{N+1}$.

Let now $x \in M_1$. Since f is surjective: $M = f^N(M)$, so there is a $y \in M$ s.t. $f^N(y) = x$.

But then $0 = f(x) = f(f^N(y)) = f^{N+1}(y) \Rightarrow y \in M_{N+1} = M_N$.

$\Rightarrow f^N(y) = 0 \Rightarrow x = 0$. $\Rightarrow f$ is injective. $\square$

(6) Let R be left noetherian and $a, b \in R$ s.t. $ab = 1$. Then $ba = 1$.

Pf: We have $R = R \circ b \subset Rb \subset R \Rightarrow R = Rb$. (3)

Thus the homomorph. $f: R \rightarrow R: x \mapsto xb$ is surjective and by (5) it is an isomorphism.

Now compute: $f(1-ba) = (1-ba)b = b - \underbrace{ba}_b = b - b = 0$

$\Rightarrow 1-ba = 0 \Leftrightarrow 1 = ba$. $\square$
 f inj.

§ 1.2. Simple + semi-simple modules

Def 1.2.1. An R -module is said to be simple if $M \neq 0$ and 0 and M are its only submodules.

M is semi-simple if it is a direct sum of simple modules.

(Thm 1.2.4: any sum of simple mods is s.s.!))

The 0 -module is def. to be semi-simple (empty direct sum).

Ex: (1) If R is any ring, then R is simple as a module over itself $\Leftrightarrow R$ is a division ring (see Lemma 1.1.1)

Any module M over a division ring is free, so every M is a semi-simple module.
only left ideals are 0 and R (lemma 1.1.2)

(2) The simple $\mathbb{Z}$ -modules are $\mathbb{Z}/p\mathbb{Z}$ for p prime.

(3) k^n considered as a column vector becomes a simple $M_n(k)$ -module. *Can be seen with Morita theory (or by matrix mult!)*
left

Remark 1.2.2

(1) If M is simple, then $M = Rx$ for every $x \neq 0$ in M .

Let $x \neq 0 \in M \Rightarrow Rx \subseteq M \xRightarrow{M \text{ simple}} Rx = M$

(2) (a) $\forall x \in M$, the map $\varphi: R/\text{ann}_R(x) \rightarrow Rx: r + \text{ann}_R(x) \mapsto rx$ is an R -module isom. (!)
 isom thm: $R \xrightarrow{\cdot x} Rx \rightarrow 0$
 $\text{ann}_R(x) = \ker(\cdot x)$

(b) M is cyclic $\Leftrightarrow M \cong R/L$ for some left ideal L in R .

($\Leftarrow$: generate: $1+L, \Rightarrow$ if $M = \langle a \rangle$, then: $R \xrightarrow{\cdot a} M \rightarrow 0 \Rightarrow M \cong R/\ker(\cdot a)$)

(c) M is simple $\Leftrightarrow M \cong R/L$ for some max. left ideal $L \subset R$. ← needed often!!
 Have bij: ideals in R/L and ideals in R contained in L

(3) (b) Every ideal is contained in a maximal ideal. (Special case of 3b)

(c) Let M be a f.g. R -module. Then every proper submodule $N \subsetneq M$ is contained in a max. submodule $N^* \subset M$ (Idea: Use Zorn's lemma for M/N).

The $\mathbb{Z}$ -module $\mathbb{Q}$ has no max. submodules.

Lemma 1.2.3: Let M be an R -module, $N \subset M$ a submodule, and $(M_i)_{i \in I}$ a family of simple submodules s.t. $M = N + \sum_{i \in I} M_i$. Then $\exists$ a subset $J \subset I$ s.t. $M = N \oplus \bigoplus_{j \in J} M_j$.

Proof: (needs Zorn's lemma!)

Consider $\Omega := \{I' \subset I : \text{the sum } N + \sum_{j \in I'} M_j \text{ is direct}\}$.

Since $\emptyset \in \Omega$, we have that $\Omega \neq \emptyset$.

First prove the

Claim A Ω is inductively ordered (w.r.t. set theoretical inclusion) is. every chain of elts in Ω has upper bound

Pf of A: Let $Y \subset \Omega$ be a non-empty totally ordered subset of Ω .

Let $I_0 = \bigcup_{I' \in Y} I'$, then $I_0 \in \Omega$. This implies that I_0 is an upper bound for Y in Ω , and Claim A follows.

(I₁, I₂) ...

(see will show!)

Now show that $I_0 \in \Omega$: have to prove that $N + \sum_{j \in I_0} M_j$ is $\oplus$.

Recall that a sum of submodules $\sum_{i \in S} L_i$ of some R -modules is direct $\Leftrightarrow$ for every finite $T \subset S$ the sum $\sum_{i \in T} L_i$ is direct. Thus: sufficient to show that $N + \sum_{j \in I} M_j$ is direct for every finite $I \subset I_0$.

Since Y is totally ordered, for every such I , there is an $I' \in Y$ s.t. $I \subset I'$. By def. of Ω , the sum $N + \sum_{j \in I'} M_j$ is direct, and hence also the smaller sum $N + \sum_{j \in I} M_j$ is $\oplus$. $\square$ (for Claim A).

Now by Zorn's lemma $\exists$ max. element $J \in \Omega$.

(5)

Claim B: $M = N \oplus \bigoplus_{j \in J} M_j$.

Pf B: By def. the sum is direct (and $\subseteq N + \sum_{i \in I} M_i$), so we only have to show:
$$M = N + \sum_{i \in I} M_i \subseteq N \oplus \bigoplus_{j \in J} M_j.$$

Assume on the contrary, that there is an $i \in I$ with $M_i \not\subseteq N + \sum_{j \in J} M_j$.

Then $i \notin J$ and $M_i \cap (N \oplus \bigoplus_{j \in J} M_j) \neq 0$.

Since by assumption M_i is simple, $(*)$ is 0, and thus the sum $N \oplus \bigoplus_{j \in J} M_j + M_i$ is direct.

$N + \sum_{j \in J \cup \{i\}} M_j \Rightarrow J \cup \{i\}$ is in Ω . $\swarrow$ To maximality of J . $\square$

Theorem 1.2.4 For an R -module M TFAE:

(a) M is semi-simple.

(b) M is a sum of simple submodules.

(c) Every submodule of M is a direct summand (i.e. $\forall N \subseteq M \exists N' \subseteq M$ s.t. $M = N \oplus N'$)

Pf: (b) $\Rightarrow$ (a): Let $(M_i)_{i \in I}$ be a family of simple submods, s.t. $M = \sum_{i \in I} M_i$. By Lemma 1.2.3 (with $N=0$) there is a $J \subseteq I$ s.t. $M = 0 \oplus \bigoplus_{j \in J} M_j$, i.e., M is s.s. (def!)

(a) $\Rightarrow$ (c): Let $(M_i)_{i \in I}$ be a family of simple submods s.t. $M = \bigoplus_{i \in I} M_i$ and let $N \subseteq M$. Then $M = N + \sum_{i \in I} M_i$. Again, by

Lemma 1.2.3. $\exists J \subseteq I$ s.t. $M = N \oplus \bigoplus_{j \in J} M_j$, i.e., N is a direct summand.

(c) $\Rightarrow$ (b): Claim: Every submodule $N' \subseteq M$ also satisfies (c).

Pf of Claim: Let $N' \subset M'$ be a submodule. Then N' is also a submodule (c) of M . Thus $\exists N'' \subset M$ with $M = N' \oplus N''$. Then $M' \stackrel{!}{=} N' \oplus (N'' \cap M')$. $\square$
 note: $M' = (N' \oplus N'') \cap M'$

Now let $M' \subset M$ be the sum of all simple submodules of M . (with $M' = 0$ if M has no simple submods). By (c), $\exists N \subset M$, s.t.
 $M = M' \oplus N$.

Have to show: $N = 0$.

Assume on contrary, that $N \neq 0$, and $\exists x \neq 0 \in N$. By Rmk 1.2.2. (3) applied to $0 \subset Rx$, R_x has a max. submodule N' .

Since $R_x \subset N$ satisfies (c) (by Claim above!), $\exists L \subset R_x$ with $R_x = N' \oplus L$. Then $L \cong (R_x)/N'$, and by maximality of N' , the module L is simple.

By def: $L \subset M'$, but on the other hand, we have $L \subset R_x \subset N$.
 acc. $M' = \sum$ of simples $\rightarrow$ to $N \cap M' = 0$. $\square$

Ex: $\mathbb{Z}/4\mathbb{Z}$ is not a semisimple $\mathbb{Z}$ -module, because $2\mathbb{Z}/4\mathbb{Z}$ is a submodule, which is not a direct summand.

(2) $k \subset k[x]$ is simple module ($k \cong k[x]/(x) \leftarrow$ max. left id.)

$\uparrow$ only simple if $k = \bar{k}$ by weak HNS, if k not alg. closed, simples are extensions of k .
 in gen: $k[x]/(p)$, p irred, are simple modules.

Cor. 1.2.5 If M is a semi-simple module, then the same is true for any submodule and every factor module.

Pf: Suppose that M is s.s., and let $N \subset M$ be a submodule.

In the proof of Thm 1.2.4 (Claim), we have seen that (c) is inherited by submods, whence N is s.s. and $\exists N' \subset M$ s.t. $M = N \oplus N'$.

We have $N' \cong M/N$, so both $N' \cong M/N$ and N are semi-simple. $\square$

Rmk 1.2.6 (1) Every simple module is noetherian and artinian

$\therefore$ does not have chains of submods: only $\{0\} \subset M$ ACC $\checkmark$ DCC $\checkmark$

every proper $N \subset M$ is cont. in max. N.S.M.

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②

($\Rightarrow$: leer)
($\Leftarrow$: leer)

Lemma 1.2.7 Let $\varphi: M \rightarrow M'$ be a nonzero R -module homom.

(1) If M is simple, then g is injective.

(2) If M' is simple, then φ is surjective.

(3) If M and M' are simple, then φ is an isomorphism

P2: Use that $\ker \varphi \subset M$ and $\text{Im}(\varphi) \subset M'$ are submodules. $\square$

e.g (1): Since M simple: $\ker \varphi = \begin{cases} 0 \Rightarrow \varphi \text{ inj} \\ M \Rightarrow \varphi = 0 \end{cases}$

(2) $\dim(Y) = \begin{cases} 0 \Rightarrow Y \equiv \emptyset \\ 1 \Rightarrow Y \text{ curve} \end{cases}$

Lemma 1.2.8 SCHUR'S LEMMA

① If M and M' are non-isom. simple R -modules, then $\text{Hom}_R(M, M') = 0$.

(2) If M is simple, then $\text{End}_R(M) = \{ \varphi: M \rightarrow M, R\text{-lin} \}, +, 0$ is a division ring.

Pr: (i): Clear by Lemma 1.2.7. $[0 \neq g \in \text{Hom}_R(M, M') \Leftrightarrow M \cong M']$

(2) By Lemma 1.2.7., each $\varphi \in \text{End}_R(M)$ is either 0 or an isom.

Then $\exists \varphi: M \rightarrow M$ s.t. $g \circ \varphi = \varphi \circ g = \text{Id}_M$. Since Id_M is the identity of $\text{End}_R M$, the claim follows. $\square$

Def 1.2.8 Let M be an R -module.

(1) Let $l(M) = l_R(M) := \sup_{l \in \mathbb{N} \cup \{\infty\}} \{ \exists \text{ chain of submods } M = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_l = 0 \}$

If $\ell < \infty$, then we say that M has finite length. (8)
 (or M is module of f. length)

(2) If $M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_\ell = 0$ (*)

with M_{i-1}/M_i simple for each $1 \leq i \leq \ell$, then (*) is called a composition series of length ℓ (for M). ($\Leftrightarrow \forall 1 \leq i \leq \ell$ there is no module M' with $M_{i-1} \supsetneq M' \supsetneq M_i$).

(b) If $M = M'_0 \supsetneq M'_1 \supsetneq \dots \supsetneq M'_k = 0$ (**)

is another chain, then (*) and (**) are equivalent if $k = \ell$ and $\exists \sigma \in S_\ell$ s.t. $M_{i-1}/M_i \cong M'_{\sigma(i)-1}/M'_{\sigma(i)}$ $\forall 1 \leq i \leq \ell$.

(c) (**) is a refinement of (*) if $\forall 1 \leq i \leq \ell$ the module $M_i \in \{M'_1, \dots, M'_k\}$.

(?)
 e.g.: $k^4 \supsetneq k^2 \supsetneq k \supsetneq 0$
 $k^4 \supsetneq k^3 \supsetneq k \supsetneq 0$
 equiv
 $\sigma = (12)$