WEEK 9

1.6.15 Term (gezodel)

Recall: We are proving: if R is a finite dim'l k-algebra and $R/J(R) \cong k^n$, then $R \cong kQ/I$ for some finite quiver Q and some admissible ideal I.

Pf: Already shown: R has complete system of orthogonal idempotents $e_1, \dots, e_n$, induced from idempotents of s-s. algebra $R/J(R)$. ← shown by induction

Now construct Q_R:

Set $(Q_R)_0 = \{1, \dots, n\}$. We have $J \overset{\text{Peirce decomp}}{\cong} \bigoplus_{i,j} e_j J e_i$ ($J = J(R)$). Then J/J^2 is both a left and a right R-module [an R-R-bimodule], so decompose as $J/J^2 = \bigoplus_{i,j \in Q_0} e_j (J/J^2) e_i = \bigoplus_{i,j} (e_j J e_i)/(e_j J^2 e_i)$.

Set now the **arrows** $i \to j$ in Q_R correspond to elements of $e_j J e_i$ inducing a k-basis of $(e_j J e_i)/(e_j J^2 e_i)$.

We get an induced homomorphism $\theta : kQ \to R$.

Claim: θ is surjective ($\overset{\text{isom. thm}}{\Rightarrow}$ $R \cong kQ/I$, where $J^m \subseteq I \subseteq J$ for some $m \geq 1$] and $\ker\theta = I \subseteq kQ$ is an admissible ideal.

Pf of claim: Let $U = \theta(kQ_+) \subseteq J$. Since $U \subseteq J \Rightarrow U + J^2 = J$ (J/J^2 = basis of U)

$\overset{\text{Nakayama Lemma}}{\Longrightarrow}$ (J superfluous) $U = J$.

This implies that $\theta : kQ \to R$ is surjective. □ claim

Set now $I := \ker\theta$. Still have to show: I is admissible.

If $m \gg 0$ such that $J^m = 0$, then $\theta((kQ_+)^m) \underset{\theta \text{ alg. hom}}{=} \theta(kQ_+)^m \overset{U = \theta(kQ_+)}{=} U^m = 0$

$\Rightarrow (kQ_+)^m \subseteq \ker\theta = I$.

For the other inclusion ($I \subseteq kQ_+^2$), let $x \in I$ ($\theta(x) = 0$). We may write

$$x = \underbrace{u}_{k\text{-lin. comb of } e_i\text{'s}} + \underbrace{v}_{k\text{-lin. comb of some arrows} \in Q_1} + w, \quad w \in kQ_+^2.$$

Since $\Theta(e_i)=e_i$ by construction and $\Theta(v), \Theta(w) \in J$

$\underbrace{\Theta(v) \in KQ_+,\ \Theta(w) \in KQ_+}_{\mathrm{Im}(KQ_+)=J}$.

we must have $\Theta(x) = \underbrace{\Theta(u)}_{\notin J \text{ (or } 0)} + \underbrace{\Theta(v)+\Theta(w)}_{\in J} = 0 \Rightarrow u=0$

(e_i K-lin. indep) induces 0-elt in J/J^2

$\Rightarrow \Theta(v) = -\underbrace{\Theta(w)}_{\in J^2} \Rightarrow \Theta(v)\in J^2$. This means that $\Theta(v)+J^2 = 0+J^2$ in J/J^2

But this means that $v=0$ (arrows are the only $\neq 0$ elts in $KQ_+/(KQ_+)^2$)

$\Rightarrow x = w \in KQ_+^2$. Thus $I \subseteq (KQ_+)^2$. □

Examples: (1) Let $A = K[x]/(x^m)$, $m \geq 2$. This is a fin. dim'l K-algebra (even comutative!) with $J(A) = (\bar{x}) = (x+(x)^m)$

- $A/J \cong K$, so we only have one idempotent e_1, so $Q_0=\{1\}$.
- $J/J^2 = (\bar{x})/(\bar{x}^2) \cong K\bar{x}$ is a 1-dim'l K-vs. $\Rightarrow$ only one arrow

$$1\bullet \circlearrowleft \alpha \qquad KQ_+ = \langle \alpha, \alpha^2, \dots, \alpha^{m-1}\rangle$$

$$\Theta: KQ \longrightarrow K[x]/(x^m)$$
$$e_1 \longmapsto 1$$
$$\alpha \longmapsto \bar{x}$$

$\Theta(KQ_+) = \langle \bar{x}, \bar{x}^2, \dots, \bar{x}^{m-1}\rangle$

$\Rightarrow \ker\Theta = \langle \alpha^m \rangle$

$$\Rightarrow K[x]/(x^m) \cong K(\bullet \circlearrowleft \alpha)/(\alpha^m).$$

(2) Let $A = \begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & K & K & 0 \\ K & K & 0 & K \end{bmatrix}$ ← fin. dim. K-algebra

$e^{ij};\ j\leq i$

Can find 4 idempotents

$e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, e_2, e_3, e_4

$\underbrace{A = \bigoplus_{i=1}^{4} Ae_i}$; so 4 idempotents are gen. A

Here $J(A) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & K & 0 & 0 \end{bmatrix}$ and $J^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ K & 0 & 0 & 0 \end{bmatrix}$

$J^3 = 0$.

Compute: $e_j J/J^2 e_i = \begin{array}{c} e_jJe_1 \cdots e_jJe_4 \\ \begin{bmatrix} 0&0&0&0\\ K&0&0&0\\ K&K&0&0\\ K&K&0&0 \end{bmatrix} \end{array} \Big/ \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ K&0&0&0\\ K&0&0&0 \end{bmatrix}$

$i \to j$

K-basis: $\begin{bmatrix} 0&0&0&0\\ K&0&0&0\\ 0&K&0&0\\ 0&K&0&0 \end{bmatrix}$ $\rightsquigarrow$ get 3 arrows: $\alpha: 1 \to 2$

$\beta: 2 \to 3$

$\gamma: 2 \to 4$

$\dim(e_j A e_i) = 1$ or 0.

So: Q:
$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\ & & 4 \end{tikzcd}$$

$\Theta: KQ \to A$
$e_i \mapsto E_i$
$\alpha \mapsto \begin{bmatrix} 0&0&0&0\\ K&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$
etc.

See here that $\ker\Theta = 0$ (e.g. $\beta\alpha \neq 0 \Rightarrow \Theta(\beta\alpha) = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&K&0&0\\ 0&0&0&0 \end{bmatrix}\begin{bmatrix} 0&0&0&0\\ K&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$
$= \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ K&0&0&0\\ 0&0&0&0 \end{bmatrix} \in J^2)$
$\neq 0$

Rmks: • One can also relate quivers to K-algebras of ∞-dimension $\rightsquigarrow$ "Ext-quiver", very useful when interested in homological properties (gldim, projectives, $D^b(\mathrm{mod}\,A)$).
In particular, to any semi-perfect noetherian ring, one can associate a quiver, as we did in the f.dim'l case. See [Hazewinkel-Gubareni-Kirichenko, "Algebras, Rings and modules", Vol. I] Chapter 11

• In rep. theory, one also associates different quivers to an algebra A, e.g. the Auslander-Reiten quiver (=AR-quiver) which encodes the structure of mod A (i.e. irred. maps between A-modules). $\rightsquigarrow$ see maybe end of the course

§2. Prime ideals

P.I. important in CA + AG: "points" of alg. varieties

2.1. Definitions and basic properties

two sided!

Def. due to Krull

Def 2.1.1 Let R be a ring.

(1) An ideal $\mathfrak{p} \subseteq R$ is called prime if • $\mathfrak{p} \neq R$, and

• Whenever $I, J \subseteq R$ are ideals with $I \cdot J \subseteq \mathfrak{p}$, then either $I \subseteq R$ or $J \subseteq \mathfrak{p}$. The set $\operatorname{Spec}(R) := \{\mathfrak{p} \subseteq R : \mathfrak{p} \text{ prime}\}$ is the spectrum of R.

(2) R is called a prime ring if $R \neq 0$ and 0 is a prime ideal (equiv: $R \neq 0$ and the product of nonzero ideals is nonzero)

In comm. alg: 0 p.i. $\Leftrightarrow$ R is a domain

Ex: Any domain is a prime ring, and any simple ring is prime (by definition).

Lemma 2.1.2 Let $\mathfrak{p} \subsetneq R$ be an ideal. TFAE:

(a) $\mathfrak{p}$ is a prime ideal.

(b) If I and J are ideals with $\mathfrak{p} \subsetneq I$ and $\mathfrak{p} \subsetneq J$, then $IJ \not\subseteq \mathfrak{p}$.

(c) $R/\mathfrak{p}$ is a prime ring.

(d) If I and J are right ideals with $IJ \subset \mathfrak{p}$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.

(e) If I and J are left ideals with $IJ \subset \mathfrak{p}$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.

(f) If $x, y \in R$ with $xRy \subset \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Pf: (a)$\Rightarrow$(b): clear (Let $\mathfrak{p} \subseteq R$ be prime. If $IJ \subset \mathfrak{p} \overset{(a)}{\Rightarrow} I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$, so $\mathfrak{p}$ cannot be strictly contained in I and J)

(b)$\Rightarrow$(c): Let I, J be ideals of $R/\mathfrak{p}$. Then there exist ideals $\mathfrak{p} \subset I' \subset R$ and $\mathfrak{p} \subset J' \subset R$ s.t. $I = I'/\mathfrak{p}$ and $J = J'/\mathfrak{p}$. If $IJ = 0$, then $I'J' \subset \mathfrak{p}$. By (b), this implies that $I' = \mathfrak{p}$ or $J' = \mathfrak{p}$, and hence either $I = 0$ or $J = 0$.

(c)$\Rightarrow$(a): Let I and J be ideals of R with $IJ \subset \mathfrak{p}$. Then $(I+\mathfrak{p})/\mathfrak{p}$ and $(J+\mathfrak{p})/\mathfrak{p}$ are ideals of $R/\mathfrak{p}$ whose product

$\Rightarrow (I+\mathfrak{p})/\mathfrak{p}=0$ or $(J+\mathfrak{p})/\mathfrak{p}=0$

is zero. Then either $I\subset\mathfrak{p}$ or $J\subset\mathfrak{p}$. left mult. (5)

(a)$\Rightarrow$(d): Let I and J be right ideals with $IJ\subset\mathfrak{p}$. Then $(RI)(RJ) = RIJ\subset\mathfrak{p}$, and hence either $RI\subset\mathfrak{p}$ or $RJ\subset\mathfrak{p}$. $\rightarrow 1\cdot I = I\subset\mathfrak{p}$

(d)$\Rightarrow$(f): Let $x,y\in R$ with $xRy\subset\mathfrak{p}$. Then $(xR)(yR)\subset\mathfrak{p}\Rightarrow xR\subset\mathfrak{p}$ or $yR\subset\mathfrak{p}$.

(f)$\Rightarrow$(a): Let $I\not\subset\mathfrak{p}$ and $J\not\subset\mathfrak{p}$ be ideals and choose $x\in I\setminus\mathfrak{p}$, $y\in J\setminus\mathfrak{p}$. Then $xRy\not\subset\mathfrak{p}\Rightarrow IJ\not\subset\mathfrak{p}$.

(a)$\Leftrightarrow$(e) by symmetry.

Lemma 2.1.3 Every maximal ideal in R is prime.
($I\subsetneq R$ and if $I\subseteq J\subseteq R$, then $J=R$ or $J=I$)

Pf: Let $\mathfrak{m}$ be a maximal ideal in R. Let I,J be ideals of R not contained in $\mathfrak{m}$. Then $I+\mathfrak{m}=R$ and $J+\mathfrak{m}=R$ and

$$R=(I+\mathfrak{m})(J+\mathfrak{m}) = IJ+\mathfrak{m}J+I\mathfrak{m}+\mathfrak{m}^2 \subset IJ+\mathfrak{m}$$

all ideals two-sided

i.e. $\mathfrak{m}+IJ=R$. Thus $IJ\not\subset\mathfrak{m}$. [Showed: $I\not\subset\mathfrak{m}, J\not\subset\mathfrak{m}\Rightarrow IJ\not\subset\mathfrak{m}$] $\square$

Lemma 2.1.4 (Let $R\neq 0$.) (1) R has a maximal ideal $\Rightarrow R$ has a prime ideal.

(2) Every prime ideal $\mathfrak{p}\subseteq R$ contains a minimal prime ideal.

(i.e. $\mathfrak{q}\subseteq\mathfrak{p}$ minimal prime $\Leftrightarrow \forall \mathfrak{q}'\subseteq\mathfrak{p}$, $\mathfrak{q}'$ prime: $\mathfrak{q}\subseteq\mathfrak{q}'$).

Pf (Use Zorn's lemma) (1) By Zorn's lemma, the set of all ideals of R containing an ideal $I\subsetneq R$ has a maximal element. (or: every ideal $I\subsetneq R$ is cont. in a max ideal)

(2) Let $\mathfrak{p}\subseteq R$ be a prime ideal and consider the set

$$\Omega=\{\mathfrak{q}\subseteq\operatorname{Spec}(R):\mathfrak{q}\subseteq\mathfrak{p}\}.$$

We show that every non-empty chain $Y\subset\Omega$ has a lower bound. Then Zorn's lemma implies that Ω has a minimal element.

So let $Y\neq\emptyset$ be a chain of prime ideals in Ω. Set $\mathfrak{q}:=\bigcap_{\mathfrak{q}'\subseteq Y}\mathfrak{q}'$.

Clearly $\mathfrak{q} \subsetneq R$ is an ideal with $\mathfrak{q} \subseteq \mathfrak{p}$. So we are finished when we show that $\mathfrak{q} \in \mathrm{Spec}(R)$. (6)

For this, let $x, y \in R$, with $xRy \subset \mathfrak{q}$ and $x \notin \mathfrak{q}$. Then $\exists$ some $\mathfrak{p}' \in Y$ with $x \notin \mathfrak{p}'$. For any $\mathfrak{p}'' \in Y$ s.t. $\mathfrak{p}'' \subseteq \mathfrak{p}'$, we have $x \notin \mathfrak{p}''$ and $xRy \subset \mathfrak{q} \subset \mathfrak{p}'' \overset{\text{Lemma 2.1.2 (f)}}{\Rightarrow} y \in \mathfrak{p}''$. In particular, $y \in \mathfrak{p}'$.

If $\mathfrak{p}'' \subset Y$ and $\mathfrak{p}'' \not\subseteq \mathfrak{p}'$, then $\mathfrak{p}' \subset \mathfrak{p}''$ and so $y \in \mathfrak{p}''$.

Thus: $y \in \mathfrak{p}''$ $\forall \mathfrak{p}'' \in Y$, and so $y \in \mathfrak{q}$, which shows that $\mathfrak{q}$ is prime.

$\Rightarrow \mathfrak{q}$ is lower bound of Y. $\square$

<u>Rmk</u> 2.1.5. Let $I \subset R$ be an ideal and $\mathfrak{p} \subseteq R$ be prime, s.t. $\mathfrak{p} \supseteq I$. Then $\mathfrak{p}/I \subset R/I$ is a prime ideal, and by Lemma 2.1.4 it contains a minimal prime $\mathfrak{q}/I \subset \mathfrak{p}/I$.

Then $\mathfrak{q} \subseteq R$ is a minimal p.i. in R containing I. We say that $\mathfrak{q}$ is a prime ideal <u>minimal over I</u>.

<u>example</u>: (1) $R = k[x,y]$ commutative. Then $0 \subsetneq \langle x \rangle \subsetneq \langle x, y \rangle$ is a chain of p.i.'s and clearly $0 \in R$ is minimal.

(2) Let $R = k[x,y]/(xy)$. Here $(\bar{0})$ is not prime, since $\bar{x} \cdot \bar{y} = \bar{0}$ in R but $\bar{x} \neq \bar{0}$ and $\bar{y} \neq 0$. Have chains: $(\bar{x}) \subsetneq (\bar{x}, \bar{y})$ and by $(\bar{x})$, $(\bar{y}) \subsetneq (\bar{x}, \bar{y})$ $(\bar{y})$ are min.

These are prime, because $R/(\bar{x}) \cong k[y]$ int.dom, $R/(\bar{x}, \bar{y}) \cong k$, field

<u>Lemma 2.1.6</u>: Let R be left noetherian (resp. right noetherian). Then R only has finitely minimal p.i.'s, and there is a finite product of minimal p.i.'s (repetition allowed) that $= 0$.

Pf. Note that in the following, we require the ACC on two-sided ideals.

It is sufficient to prove that $\exists$ prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subsetneq R$ s.t.

$\mathfrak{p}_1 \cdots \mathfrak{p}_n = 0$. ($\because$ To see this: note that after replacing each $\mathfrak{p}_i$ by a minimal prime ideal contained in it, we may assume that each $\mathfrak{p}_i$ is minimal. Now let $\mathfrak{p}$ be any minimal p.i.. Then $\mathfrak{p}_1 \cdots \mathfrak{p}_n = 0 \subseteq \mathfrak{p}$. Thus $\exists j \in \{1,\dots,n\}$ s.t. $\mathfrak{p}_j \subseteq \mathfrak{p}$, so by minimality: $\mathfrak{p} = \mathfrak{p}_j$.
Thus the minimal prime ideals of R are contained in the finite set $\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$.)

Assume now on the contrary that no finite product of p.i.'s is $=0$ and consider the set

$$\Omega = \{I \subset R : I \text{ is an ideal not containing a product of prime ideals}\}.$$

Then $0 \in \Omega$ (so $\Omega \neq \emptyset$), and by Noetherian hypothesis, Ω contains a maximal element K. Since R/K is a counterexample to the theorem, we may replace R by R/K. Thus we may assume that no finite product of p.i.'s in R is zero, while all non-zero ideals in R contain finite products of p.i.'s.

In particular, 0 cannot be a prime ideal. Hence, $\exists$ nonzero ideals $I, J \subseteq R$, s.t. $I \cdot J = 0$. Then there are p.i.'s $\mathfrak{p}_1,\dots,\mathfrak{p}_m$, $\mathfrak{q}_1,\dots,\mathfrak{q}_n$ in R with $\mathfrak{p}_1 \cdots \mathfrak{p}_m \subset I$ and $\mathfrak{q}_1 \cdots \mathfrak{q}_m \subseteq J$.
However, this implies that $\mathfrak{p}_1 \cdots \mathfrak{p}_m \mathfrak{q}_1 \cdots \mathfrak{q}_n \subseteq IJ = 0$. $\lightning$ $\square$

Rmk Passing from R to R/K is known as Noetherian induction. Since R/K is the smallest factor ring of R violating the theorem, it is known as minimal criminal.

Def 2.1.7 A nonempty set S of a semigroup H is said to be an m-system if, for any $a, b \in S$, $\exists h \in H$ s.t. $ahb \in S$.
$\leftarrow$ general. of multiplicatively closed set ($S \subseteq R$ mult. $\Leftrightarrow$: $\cdot\, 1_R \in S$, $\cdot\, \forall s, s' \in S : s \cdot s' \in S$)

Rmk 2.1.8 (1) Every multiplicatively closed subset is an m-system. The set $\{a, a^2, a^4, a^8, \dots\}$ for $a \in \mathbb{Z}_{>1}$ is an m-system that is not mult. closed.

(2) An ideal $\mathfrak{p} \subset R$ is a prime ideal $\iff R \setminus \mathfrak{p}$ is an m-system (see Lemma 2.1.2.(f)). In the context of rings, m-systems are introduced in [Lam, ...].

<u>Lemma 2.1.9</u> (Krull's existence thm for non-commutative rings)
Let $I \subseteq R$ be an ideal, $T \subset R$ be an m-system with $I \cap T = \emptyset$, and $\Omega = \{ J \subset R : J$ is a two-sided ideal with $I \subset J$ and $J \cap T = \emptyset \}$

If R comm. and T mult. closed, then $\Omega \xleftrightarrow{1-1} \{$prime ideal in localization $S^{-1}R\}$

Then: (1) Ω has a maximal element.

(2) Every maximal element of Ω is a prime ideal.

<u>Pf</u> (1) Let $\Sigma \subset \Omega$ be a chain. Then $\bigcup_{\mathfrak{p} \in \Sigma} \mathfrak{p}$ is an upper bound for Σ.

Thus Ω is inductively ordered, and has a maximal element by Zorn's lemma.

(2) Let $\mathfrak{p} \in \Omega$ be maximal, and assume on the contrary that $\exists$ $a, b \in R \setminus \mathfrak{p}$ with $aRb \subset \mathfrak{p}$. By the maximality of $\mathfrak{p}$, there are $s, s' \in T$ with $s \in \mathfrak{p} + RaR$ and $s' \in \mathfrak{p} + RbR$. If $r \in R$ with $srs' \in T$, then $srs' \in (\mathfrak{p} + RaR)R(\mathfrak{p} + RbR) \subset R\underbrace{aRb}_{\in \mathfrak{p}}R \subset \mathfrak{p}$ ↯ $\square$