WEEK 8

Recall : KQ = path algebra of quiver

modules over KQ $\overset{1:1}{\longleftrightarrow}$ representations of Q

$V \mapsto (e_i V$, maps $V_a : V_i \to V_j$ given as $a : i \to j$ for $a \in e_j V e_i$)

$(V = \bigoplus_{i \in Q_0} V_i$, with action $e_i v = v_i$ and $a_1 \cdots a_n v = V_{a_1} \circ V_{a_2} \circ \cdots V_{a_n}(v_{t(a_n)}) \in V_{h(a_1)}$) $\longleftarrow$ $(V_i, V_a : V_i \to V_j)$

Now we are approaching a theorem of P. Gabriel, that essentially says that any f.dim'l K-algebra A (with some restriction depending on $A/J(A)$) will be of the form KQ/I. $\leftarrow$ admissible ideal

Notation 1.6.6: (a) Denote by $(KQ)_+$ the K-span of the nontrivial paths. This is an ideal (clear: mult increases length) and $(KQ)/(KQ)_+ \cong K^{(Q_0)}$. $\leftarrow$ also clear $KQ/KQ_+ = \langle e_1, \ldots, e_n \rangle$

(b) We write $P[i]$ for the KQ-module KQe_i (paths starting at i), so $KQ = \bigoplus_{i \in Q_0} P[i]$.

Considered as a representation of Q, the vector space at vertex j has as basis the paths from i to j.

e.g. $Q : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$

$P[1] : K \xrightarrow{\alpha} K \xrightarrow{\beta} K$

$P[2] : 0 \to K \xrightarrow{\beta} K$

$P[3] : 0 \to 0 \to K$

written as modules: $P[1] = {}_K\langle e_1, \alpha, \alpha\beta\rangle$ $P[2] = {}_K\langle e_2, \beta\rangle$
$P[3] = {}_K\langle e_3\rangle$ $KQ \cong$ 6-dim'l K-algebra

$P[i]$ are the projectives of KQ.

(c) We write $S[i]$ for the representation with $S[i]_i = K$ and $S[i]_j = 0$ for $j \neq i$ and all $S[i]_a = 0$. It corresponds to the module $KQe_i/(KQ)_+e_i$ ← we will see that these are the simple KQ-modules.

For the rest of this section we will assume that K = field.

<u>Example</u>: $Q = 1 \to 2$ $P[1]: K \xrightarrow{1} K$ $S[1] = K \xrightarrow{0} 0$
$P[2] = S[2] = 0 \xrightarrow{0} K$

- Compute $\mathrm{Hom}(S[1], P[1]) = 0$ $\mathrm{Hom}(S[2], P[1]) = K$.
- The subspaces $(K \le V_1, 0 \le V_2)$ does not give a subspace of $P[1]$. But $(0 \le V_1, K \le V_2)$ is a subrep of $P[1]$ and it is $\cong$ to $S[2]$.
- There is a s.e.s. $0 \to S[2] \to P[1] \to S[1] \to 0$.
- $S[1] \oplus S[2] \cong K \xrightarrow{0} K$ and for $\lambda \neq 0 \in K$ we have $K \xrightarrow{\lambda} K \cong P[1]$.
- Every representation of Q is isomorphic to a direct sum of copies of $S[1]$, $P[1]$, and $S[2]$: For a f.dim'l rep. $V_1 \xrightarrow{a} V_2$ one can see this as follows: take bases of V_1, V_2, then the rep. is isom to $K^n \xrightarrow{A} K^{m}$ n for some $m \times n$ matrix A. ~ may write as $K^n \xrightarrow{A} K^n$ $n \geq m$ with 0s. Now Gauss-elim. tells us that $\exists P, Q$ invertible s.t. $PAQ^{-1} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, with I = id. matrix. Then get isom. of reps
$$\begin{array}{ccc} K^n & \xrightarrow{A} & K^m \\ Q\downarrow & & \downarrow P \\ K^n & \xrightarrow{PAQ^{-1}} & K^{m}\ n \end{array}$$
, and the last rep. is a direct sum as claimed.

<u>Lemma 1.6.7</u> (i) The $P[i]$ are non-isom. indec. KQ-modules
(ii) The $S[i]$ are non-isom. simple KQ-modules
(iii) If $|Q_0| < \infty$ and contains no oriented cycles, ← i.e. KQ is fin. dim'l K-algebra! Lemma 1.6.4! then (KQ_+) is nilpotent, so it is the Jacobson radical of KQ and the $S[i]$ are all of the

simple modules.

Pf: (i) The spaces $P[i] = KQe_i$, $e_j KQ e_i$, and $e_j KQ$ have as K-bases the paths with tail at i, with tail at i and head at j, and with head at j.

If $f \neq 0 \in KQe_i$ and $g \neq 0 \in e_i KQ$, then $fg \neq 0$: explicitly, if p and q are paths of max. length involved in f and g, then the coefficient of pq in fg must be $\neq 0$. $eAe \cong \mathrm{End}_A(Ae)^{op}$

Now by $\boxed{\text{Lemma 2}}$: $\mathrm{End}_{KQ}(P[i])^{op} = \mathrm{End}_{KQ}(KQe_i, KQe_i)^{op} \cong e_i KQ e_i$ (as rings) and by the observation above this is a domain (products of $\neq 0$ elements are non-zero). This means that $e_i KQ e_i$ has no nontrivial idempotents $\overset{\text{Lemma 3}}{\Longrightarrow}$ $P[i]$ is indecomposable.

(assume $e \in A$ is a domain and has non-trivial idempotent $e \Rightarrow A = Ae \oplus A(1-e)$ then $\underset{\neq 0}{e} \in Ae$ and $(\underset{\neq 0}{1-e}) \in A(1-e) \Rightarrow \underset{\neq 0}{e}(\underset{\neq 0}{1-e}) = e - e^2 = 0 \Rightarrow A$ not dom.)

If $P[i] \cong P[j]$, then there are inverse isomorph. between the two modules, and so for an element $f \in e_j KQ e_i$, and element $g \in e_i KQ e_j$ we have $fg = e_j$ and $gf = e_i$. Then, by argument above (product of nontrivial paths cannot be trivial!) f and g can only involve trivial paths, so $i = j$.

(ii) Clear (there are no nonzero proper submodules of K!)

(iii) Since $|Q_0| < \infty$ and Q contains no oriented cycles, there is a bound on the length of any path. So $(KQ)_+$ is nilpotent.

This means $J(KQ) \supseteq (KQ)_+$.

Since $KQ/(KQ)_+ \cong K^{(Q_0)}$, it is semi-simple [direct product of simple module K]

This means $(KQ)_+ \supseteq J(KQ)$. $\Rightarrow J(KQ) = (KQ)_+$.

The simples are indexed by Q_0 ($\rightarrow KQ/(KQ)_+ \cong K^{|Q_0|}$), so there are no other simples than the $S[i]$. $\square$ (Altern: [ASS I.1.4 (c): $A/I \cong K^n \Rightarrow I = \mathrm{rad} A$)

Def 1.6.8: Suppose that $|Q_0| < \infty$. An ideal $I \subseteq KQ$ is <u>admissible</u> if

(1) $I \subseteq (KQ)_+^2$ and
(2) $(KQ)_+^n \subseteq I$ for some n.

The pair (Q,I) is the called a <u>bound quiver</u> and KQ/I a <u>bound quiver algebra</u>.

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Gabriel thm.: KQ/I f.d. = Prop 2.6 [ASS]
$J(KQ/I) = J(KQ)/I$ Lemma 2.10
KQ/I ~~and~~ basic.

<u>Examples 1.6.9</u>: (1) If Q has no oriented cycles, then $I=0$ is admissible ($0 \subseteq (KQ)_+^2$ and $(KQ)_+$ nilpotent by Lemma 1.6.7. (iii))

(2) Let Q be the quiver $\begin{tikzcd} 1 \arrow[loop right, "x"] \end{tikzcd}$, then $KQ = K[x]$. The admissible ideals in KQ are of the form (x^n), $n \geq 2$. (Q not acyclic, so $KQ_+ \neq J(KQ)$ in this case)
0

(3) Let Q be the quiver
$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "b"'] & 2 \arrow[d, "c"] \\ 3 \arrow[r, "d"'] & 4 \end{tikzcd}$$
and let I be the ideal $I = (ca - db)$

Here $KQ_+ = (a,b,c,d,ca,db)$
$J(KQ)^2 = (ca, db)$
$J(KQ)^3 = 0$

$\Rightarrow \underbrace{J(KQ)^3}_{0} \subseteq I \subseteq J(KQ)^2$.
and I is admissible.

<u>Lemma 1.6.10</u> Let Q be a finite quiver and $I \subseteq KQ$ be an admissible ideal. Then the $(\varepsilon_i = e_i + I)_{i \in Q_0}$ form a complete set of primitive orthogonal idempotents of KQ/I.

<u>Pf</u>: Since the ε_i are the image of the orth. idempotents e_i under the projection $KQ \to KQ/I$, complete + orth. follows.
For primitive note that e_i prim $\Leftrightarrow e_i R e_i \cong \mathrm{End}_R(Re_i)$ local (*)
te <u>e_i are primitive</u> (Show e_i are only non-triv. idempotent in $e_i KQ/I e_i$: and 0 If v is another idemp., then if is of the form $v = \lambda e_i + w + I$, where $\lambda \in K$ $w \in e_i KQ/I e_i$ is a lin comb of cycles th. i of length ≥ 1

(*) Criterion is proven in [Schiffler] Cor. 4.13

Then $0 \overset{KQ/I}{=} v^2 - v = (\lambda e_i + w)(\lambda e_i + w) - (\lambda e_i + w) =$
$= \lambda^2 e_i + \lambda w + \lambda w + w^2 - \lambda e_i - w = (\lambda^2 - \lambda)e_i + (2\lambda - 1)w + w^2$
i.e. $(\lambda^2-\lambda)e_i + (2\lambda-1)w + w^2 \in I$. Since $I \subseteq KQ_+^2$: $\lambda^2 - \lambda = 0$, i.e. $\lambda = 0$ or $\lambda = 1$.
• $\lambda = 0$, then $v = w + I$. But since $I \supseteq KQ_+^m \Rightarrow w^m \in I$, so $w^m = 0$ in KQ/I. So $w \in I$ and $v = 0$.

- If $\lambda = 1$, then $v = e_i + w + I$, so $e_i - v = -w + I$ is also idempotent in $e_i(KQ/I)e_i$
$\Rightarrow w$ idemp. mod $I \underset{\text{similar as before}}{\Longrightarrow} e_i = v$.

Now a similar argument shows that the ε_i are primitive in KQ/I. [see Lemma II.2.4 in [ASS]] $\square$

Prop 1.6.11: Let Q be a finite quiver and I an admissible ideal. Then $R = KQ/I$ is a finite dim'l K-algebra.

Pf: Since I is admissible, $\exists m \geq 2$ s.t. $(KQ)_+^m \subseteq I$. But then there is a surj. algebra homom: $KQ/(KQ)_+^m \twoheadrightarrow KQ/I$. Thus it is enough to prove that $KQ/(KQ)_+^m$ is finite dim'l. Now the residue classes of paths of length $< m$ form a K-basis of $KQ/(KQ)_+^m$. There are only fin. many of these $\Rightarrow KQ/(KQ)_+^m$ is fin. dim'l.

Ex 1.6.12 (Dieudonné): If I is not admissible, then KQ/I may not be f. gen. Let Q: $\begin{tikzcd} \bullet \arrow[loop left, "\beta"] \arrow[loop right, "\alpha"] \end{tikzcd}$ and $I = \langle \beta^2, \alpha\beta \rangle$. I not adm., because $\alpha^m \notin I$ for any $m \geq 1$, but $\alpha^m \in (KQ)_+^m = \langle \alpha^m, \beta^m, \alpha^{m-1}\beta, \dots \rangle$
KQ/I is not fi. dim $\rightsquigarrow$ cf. ex. II.2.5 in [ASS].

Lemma 1.6.13 Let KQ be a finite quiver. Then every admissible ideal I is f. gen. In part. the generators are of the form $\{\rho_1, \dots, \rho_n\}$, where ρ is a relation: $\rho = \sum_{i=1}^{m} \lambda_i w_i$, $\lambda_i \in K$, where w_i are paths of length ≥ 2 s.t. if $i \neq j$, then the source (resp. target) of w_i coincides with that of w_j).

e.g.: $\begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[d, "d"'] \arrow[drr, "c"] & & 3 \\ 4 \arrow[rr, "e"'] & & 5 \arrow[u, "f"'] \end{tikzcd}$

Relation: $\rho = ba + fc + fed$

Pf: See Lemma II.2.8 + Cor II.2.9 of [ASS]. $\square$

Lemma 1.6.14 Let Q be a finite quiver an I be an admissible ideal $\subseteq KQ$ and let $R = KQ/I$. Then $J(R) = (KQ)_+/I$ and $R/J(R) \simeq K^{|Q_0|}$.

Pf: Since I is admissible, $\exists m \geq 2$, s.t. $J(R)^m \subseteq I \Rightarrow$

$\mathcal{J}(KQ/I)^m = 0$ and $\mathcal{J}(KQ/I)$ is an nilpotent ideal in R.

On the other hand $\underbrace{(KQ/I)}_{R} / \underbrace{(KQ)_+/I}_{\mathcal{J}(R)} \underset{\text{isom thm}}{\cong} KQ/(KQ)_+ \cong K^{|Q_0|}$. $\square$

$\Rightarrow$ assertion (again in Lemma I.1.4 [ASS])

Cor 1.6.15: For any $\ell \geq 1$ we have $\mathcal{J}(KQ/I)^\ell = ((KQ)_+/I)^\ell$.

Pf: Look at $\mathcal{J}(KQ/I)/\mathcal{J}((KQ)/I)^2 \cong (KQ)_+/I \big/ ((KQ)_+/I)^2$

$\cong (KQ)_+/(KQ)_+^2$ as K-vs.

as a basis we have $\bar{\alpha} + \mathcal{J}(KQ/I)^2$, where $\bar{\alpha} = \alpha + KQ/I$ and $\alpha \in Q_1$ is an arrow. $\square$

Theorem (Gabriel) (1) If I is an admissible ideal in KQ, then $R = KQ/I$ is finite dimensional K-algebra, $\mathcal{J}(R) = (KQ)_+/I$ and $R/\mathcal{J}(R) \cong K \times \dots \times K$.

(2) Conversely, if R is a finite dim'l K-algebra and $R/\mathcal{J}(R) \cong K \times \dots \times K$ then $R \cong KQ/I$ for some finite quiver Q and some admissible ideal I.

Pf: (1) Follows from Prop 1.6.11 and Lemma 1.6.15.

(2) First note that for any idempotent $e \in R$ (K-alg.) and M an R-module and $N \subseteq M$ submodule, we have $e(M/N) \underset{K\text{-mods.}}{\cong} eM/eN$.

We sketch this direction, complete argument in [ASS, II.3]: The idea is to construct a quiver Q_R for R and then show that $R \cong KQ_R/I$ as K-algebras. Construction: know that $R/\mathcal{J}(R)$ is s.s. $\rightsquigarrow$ basis of orth. idempotents $\rightsquigarrow$ "lift" idempotents to R (Nakayama Lemma!) $\rightsquigarrow$ vertices of Q_R

$\rightsquigarrow$ arrows = "paths of length 1 in KQ" = "$(KQ_+)/(KQ)_+^2 = \mathcal{J}/\mathcal{J}^2$"

More detailed: Let $J = J(R)$ be the Jacobson radical of R. (7)

By our assumption $\underset{S}{\overset{\shortparallel}{R/J}} \cong \underbrace{K\times\dots\times K}_{n\text{-factors}}$, so S has a basis

of orthogonal idempotents $f_i = (0,\dots,0,1,0,\dots,0)$, $i\in\{1,\dots,n\}$.

Set $f := \sum_{i=1}^{n-1} f_i = (1,\dots,1,0)$, then $fSf = \begin{pmatrix}1&&&\\&\ddots&&\\&&1&\\&&&0\end{pmatrix}\begin{pmatrix}1&&\\&\ddots&\\&&1\end{pmatrix}\begin{pmatrix}1&&&\\&\ddots&&\\&&1&\\&&&0\end{pmatrix} = \left(\begin{array}{ccc|c}1&&&0\\&\ddots&&\\&&1&\\\hline 0&&&0\end{array}\right)\left(\begin{array}{ccc|c}1&&&0\\&\ddots&&\\&&1&\\\hline 0&&&0\end{array}\right)$

$= \left(\begin{array}{c|c}1_{n-1}&0\\\hline 0&0\end{array}\right) \cong K^{n-1} = \{(a_1,\dots,a_{n-1},0)\}$

Now we show by induction on n that if $R/J\cong K^n$, then there are orthogonal idempotents e_i in R lifting the idempotents f_i in R/J. Let e be a lift of f, i.e., $e\in R$ idempotent, such that $f = e + J\in R/J$ and $f - e\in J$. Such a lift exists for any idempotent of a K-alg. $A/J(A)$, where A is a f. dim'l K-algebra. [see ASS, Lemma I.4.4. p19]:

Let $\eta = g + J(A)\in A/J(A)$, $g\in A$, be an idempotent, then $\exists$ idemp. $e\in A$, s.t. $e - g\in J(A)$: Pf: We know $J(A)^m = 0$ for some $m>1$. Since $\eta^2=\eta$, we have $(g+J(A))(g+J(A)) = g^2 + J(A) = g + J(A)$ and thus $(g^2-g)\in J(A)$.

$\Rightarrow (g-g^2)^m = 0 = g^m - g^{m+1}\underbrace{\left(\sum_{j=1}^{m}(-1)^{j-1}\binom{m}{j}g^{j-1}\right)}_{=:t}$

$\overset{(i)}{\Rightarrow} g^m = g^{m+1}\cdot t$ and $\overset{(ii)}{} gt = tg$ (computation!)

Claim: $e := (gt)^m$ is the idempotent lifting η.

Pf Claim: $e = g^m t^m \overset{(i)}{=} (g^{m+1}t)t^m = g^{m+1}t^{m+1} = \dots = g^{2m}t^{2m} = ((gt)^m)^2 = e^2$

so e is idempotent.

Next: $\overset{(iii)}{} g - g^m \in J(A)$ because $g - g^2\in J(A)$ implies:

$g - g^m = g(1-g^{m-1}) = g(1-g)(1+\dots+g^{m-2}) = (g-g^2)(1+\dots+g^{m-2}) \in J(A)$ ✓

Further: $g - gt\in J(A)$ because

$g + J(A) \overset{(iii)}{=} g^m + J(A) \overset{(i)}{=} g^{m+1}t + J(A) = (g^{m+1}+J(A))(t+J(A)) \overset{iii}{=}$

$(g^m + J(A))(g+J(A))(t+J(A)) \overset{iii}{=} (g+J(A))(g+J(A))(t+J(A)) = (g^2+J(A))(t+J(A))$

$\overset{g^2-g\in J}{=} (g+J(A))(t+J(A)) = gt + J(A)$. ✓

Now: $e + J(A) = g^m t^m + J(A) \overset{(ii)}{=} (gt + J(A))^m = (g+J(A))^m = g^m + J(A) = g + J(A)$

and $\Rightarrow e-g \in J(A)$.] $\square$

If e is the lift of f in R, then eJe is a nilpotent ideal in eRe.

[$(eJe)^2 = eJeeJe = eJeJe \overset{J \text{ two-sided id.}}{\subseteq} eJ^2e \overset{\text{induction}}{\rightsquigarrow} (eJe)^m \subseteq eJ^me = 0$.]

$\overset{\text{Lemma 1.4.6}}{\Longrightarrow} eJe \subseteq J(eRe)$

$eRe/\underset{e(M/N)=eM/eN}{eJe} \overset{!}{\cong} e\underbrace{(R/J)}_{S}e \overset{e \text{ lifts } f}{=} f(R/J)f \cong K^{n-1}$. ← Use again [ASS I.1.4]

Thus $J(eRe) = eJe$, and by induction there are idempotents $e_1,\dots,e_{n-1}$ in eRe inducing the idempotents $f_1,\dots,f_{n-1}$ in R/J. Then take $e_n = 1-e$.

Now construct Q_R:

Set $(Q_R)_0 = \{1,\dots,n\}$. We have $J \overset{\text{Peirce decomp}}{=} \bigoplus_{i,j} e_jJe_i$. Then J/J^2 is both a left and a right R-module [an R-R-bimodule], so decompose as
$$J/J^2 = \bigoplus_{i,j\in Q_0} e_j(J/J^2)e_i = \bigoplus_{i,j} (e_jJe_i)/(e_jJ^2e_i).$$

Set now the arrows $i \to j$ in Q_R correspond to elements of e_jJe_i inducing a K-basis of $(e_jJe_i)/(e_jJ^2e_i)$.

We get a homomorphism $\Theta: KQ \to R$.