WEEK 7

<u>Ex. 1.5.7</u> (Group algebras) Let K be a commutative ring and $(G, \cdot)$ be a (semi-)group. The (semi-)group algebra $K[G]$ (or KG) is a free K-module with basis $g \in G$ (sometimes written e_g) with mult: $\left(\sum_{g\in G} a_g e_g\right)\left(\sum_{g\in G} b_g e_g\right) := \sum_{g\in G}\left(\sum_{\substack{g_1,g_2\in G\\ g_1\cdot g_2=g}} a_{g_1} b_{g_2}\right) e_g$

The map $K \to KG$, $a \mapsto a e_{1_G}$ $\forall a \in K$ is a ring monom. and hence we may consider K as a subring of KG. Since $K \subset Z(KG)$,

$a\in K: a\cdot \sum_{g\in G} a_g g = a \cdot 1_G \cdot \sum_{g\in G} a_g g$

$= \sum_{g\in G} (a a_g) g$ $\xrightarrow{\quad}$ since K = comm.

$\quad\quad\quad\quad \|$

$\left(\sum_{g\in G} a_g g\right)\cdot(a\cdot 1_G) = \sum_{g\in G}(a_g a)\, g \cdot 1_G$

KG is a K-algebra (see $\underbrace{\text{Rmk } 1.5.2.(2)}_{\text{central ring hom } K\to KG}$). The map $G \to KG$, $g \mapsto e_g$, $\forall g \in G$ is a (semi-)group monomorphism and hence we may consider G as a sub(semi-)group of KG.

<u>Thm 1.5.8</u> (Maschke) Let K be a field and G be a finite group. Then KG is semi-simple $\iff$ char $K \nmid |G|$.

Pf: See Presentation.

Interlude: Indecomposable modules and idempotents for K-algebras

Let K be a commutative ring and let A be a K-algebra.

Recall that $e \in A$ is <u>idempotent</u> if $e^2 = e$; e_i, e_j are <u>orthogonal</u> if $e_i e_j = 0$ for $i \neq j$; and $e_1, \dots, e_n$ is <u>complete</u> if $1 = e_1 + \dots + e_n$.

<u>Lemma 1</u> If M is a left A-module, then

(i) If e is idempotent, then $eM = \{em : em = m\}$ is a K-submodule of M. (usually not an A-submod! e.g. $\overset{A = M_2(K)}{e_1 A = \left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)}$ is not left A-mod!)

(ii) If (e_i) are orthogonal idempotents, then the sum $\sum_{i \in I} e_i M$ is direct.

(iii) If $e_1, \dots, e_n$ is a complete family of orthogonal idempotents, then $M = e_1 M \oplus \dots \oplus e_n M$.

Pf: All straightforward, e.g. (i) if $em = m$, then $m \in eM$, while if $m \in eM$, then $m = em' = e^2 m' = e(em') = em$. K-submod clear! □

<u>Prop</u> (Peirce decomposition) If $e_1, \dots, e_n$ is a complete family of orthogonal idempotents, then $A = \bigoplus_{i,j=1}^{n} e_i A e_j$.

Write as a matrix:

$$A = \begin{bmatrix} e_1 A e_1 & e_1 A e_2 & \cdots & e_1 A e_n \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ e_n A e_1 & \cdots & \cdots & e_n A e_n \end{bmatrix}$$

then mult. in A acc. to matrix multiplication.

<u>Rmk</u> If e is an idempotent, then eAe is an algebra with the same operations as A, with unit element e. Since the unit is not the same as for A, it is not an A-subalgebra. Sometimes eAe is called the corner algebra of A.

e.g.: $A = M_2(K)$, $e = e_2$, then $e_2 A e_2 = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \cong K$

Lemma 2 For M a left A-module, we have $\mathrm{Hom}_A(A,M)\cong M$ as A-modules and if $e\in A$ is idempotent, then $\mathrm{Hom}_A(\overset{\text{left }A\text{-mod}}{Ae},M)\cong eM$ as k-submodules. In particular $A\cong \mathrm{End}_A(A)^{op}$ (if we used right modules, then we would not have to use the op here!) and $eAe\cong\mathrm{End}_A(Ae)^{op}$.

Pf: Send $\underset{\mathrm{Hom}_A(A,M)}{\overset{}{\underset{\cup}{\Theta: A\to M}}}$ to $\Theta(1)$ and $m\in M$ to $a\mapsto am$, etc.

$\Phi:\mathrm{Hom}_A(A,M)\to M$ $\Theta\longmapsto\Theta(1)$ ψ,Φ $\overset{A\text{-alg.}}{\text{hom}}$ ✓

$\psi: M\to\mathrm{Hom}_A(A,M)$, $m\mapsto(a\mapsto am)$

$\Phi\circ\psi(m)=\Phi(a\mapsto am)\overset{(a)}{=}1\cdot m=m$

$\psi\circ\Phi(\Theta)=\psi(\Theta(1))\overset{(b)}{=}(a\mapsto\underbrace{a\Theta(1)}_{\Theta(a)})$ A-alg hom $=\Theta(a)$ $\forall a\in A$

[detailed proof for right modules, see [ASS, Lemma 4.2]

Assem-Simson-Skowroński: "Elements of the representation theory of associative algebras" I

Decompositions

We have alread seen decomp. of $\mathrm{Hom}_R(\oplus_i M_i,\oplus_j N_j)$ in Lemma 1.3.12 ⇝ can do the same for k-algebras A, and A-modules.

In part: For the algebra $\mathrm{End}_A(M_1\oplus\ldots\oplus M_n)$, the projections onto the factors M_i give a complete family of orth. idempotents, and the Pierce decomposition is

$$\mathrm{End}_A(M_1\oplus\ldots\oplus M_n)=\begin{bmatrix}\mathrm{Hom}(M_1,M_1) & \cdots & \mathrm{Hom}(M_n,M_1)\\ \vdots & \ddots & \\ \mathrm{Hom}(M_1,M_2) & & \mathrm{Hom}(M_n,M_n)\end{bmatrix}$$

and $\mathrm{End}_A(M^n)=M_n(\mathrm{End}_A(M))$.

Lemma 3 An A-module M is indecomposable (↗ if $M=X\oplus Y$, then $X=0$ or $Y=0$) $\Longleftrightarrow$ $\mathrm{End}_A(M)$ has no nontrivial idempotents (other than 0 and 1).

ASS, cor 4.8

Pf: $\Leftarrow$: Assume $M=X_1\oplus X_2$, then $\exists\pi_i:M\to X_i$ and $\exists u_i:X_i\hookrightarrow M$ s.t. $u_1\pi_1+u_2\pi_2=\mathrm{Id}_M$ but u_ip_i are $\neq0$, idempotents in $\mathrm{End}_A M$.

$\Longrightarrow$: Let M be (f.dim'l) & indec., assume $\exists$ nonzero idemp. e_1, $e_2=1-e_1\Rightarrow M\cong\mathrm{Im}(e_1)\oplus\mathrm{Im}(e_2)$

$e_i: M\to M$, $m\mapsto me_i$

§1.6: Extended example of construction of K-algebras: Path algebras

Literature: [ASS], [ARS = Auslander-Reiten-Smalø]
[Schiffler "Quiver Representations"]

Consider K-algebras, where K is a comm. ring.

Def 1.6.1: A quiver is a quadruple $Q=(Q_0, Q_1, h, t)$, where Q_0 is a set of vertices, Q_1 is a set of arrows, and $h,t : Q_1 \to Q_2$ are mappings spezifying the head and tail vertices of each arrow.

$$\overset{t(a)}{\bullet} \xrightarrow{a} \overset{h(a)}{\bullet}$$

A path in Q of length $n > 0$ is a sequence $p = a_1 a_2 \cdots a_n$ of arrows satisfying $t(a_i) = h(a_{i+1})$ $\forall i \in \{1, \dots, n\}$

$$\bullet \xleftarrow{a_1} \bullet \xleftarrow{a_2} \bullet \xleftarrow{a_3} \bullet \cdots \bullet \xleftarrow{a_n}$$

or $\xrightarrow{a_1} \xrightarrow{a_2} \cdots \xrightarrow{a_n}$ $t(a_{i+1}) = h(a_i)$ $p = a_n \cdots a_2 a_1$

Caution: a lot of literature uses the convention $p = a_1 a_2 \cdots a_n$ for $\bullet \xrightarrow{a_1} \xrightarrow{a_2} \cdots \bullet \xrightarrow{a_n} \bullet$ i.e. $h(a_i) = t(a_{i+1})$!

Our convention is for working with left modules and paths are written like composition of functions: $a_1 a_2$ means "first a_2, then a_1"

The head and tail of p are $h(a_1)$ and $t(a_n)$.

$$p = \underset{t(p)}{\bullet} \xrightarrow{a_n} \to \longrightarrow \xrightarrow{a_2} \xrightarrow{a_1} \underset{h(p)}{\bullet}$$

For each vertex $i \in Q_0$ there is a trivial path e_i of length 0 with $h(e_i) = t(e_i) = i$.

Now write KQ for the free K-module with basis all paths in Q. It has a multiplication of paths:

$$p \cdot q = \begin{cases} 0 & \text{if } t(p) \neq h(q) \\ pq, & \text{the concatenation of } p \text{ and } q \text{ otherwise} \end{cases}$$

explicitly $p = a_1 \cdots a_n$ $q = b_1 \cdots b_m$

$$\xleftarrow{a_1} \cdots \xleftarrow{a_n} \bullet \xleftarrow{b_1} \bullet \cdots \xleftarrow{b_m} \bullet$$

$$\uparrow$$

$pq = a_1 \cdots a_n b_1 \cdots b_m$

In the following, we will always assume that Q is <u>finite</u>, i.e. $|Q_0| < \infty$. If not finite: then $\nexists$ identity $\rightsquigarrow KQ$ not unital
see Lemma 1.6.3

<u>Rmk 1.6.2</u>: A multiplication on KQ is given as follows: assume that $(a_j)_{j\in I}$ is a o.s. basis of KQ. Take $a, a' \in kQ$:

$$a = \sum_{j\in I} \lambda_j a_j \;,\quad a' = \sum_{j'\in I} \lambda_{j'} a_{j'} \;\Longrightarrow\; aa' = \sum_{j,j'\in I} \underbrace{\lambda_j \lambda_j}_{\text{mult. in } K} \underbrace{(a_j a_{j'})}_{\text{concat. if } t(a_j) = h(a_{j'})}$$

This mult. makes KQ a K-algebra, the <u>path algebra of the quiver Q</u>.

<u>Lemma 1.6.3</u>: In a path algebra KQ, the identity element is given by the sum of all constant paths: $1_{KQ} = \sum_{i\in Q_0} e_i$.

<u>Pf</u>: Let $a \in \underset{K\text{-basis } (a_j)_{j\in I}}{\underset{\swarrow}{KQ}}$, then $a = \sum_{j\in I} \lambda_j a_j$ with $\lambda_i \in K$. Then

$$\sum_{i\in Q_0} e_i \cdot a = \sum_{i\in Q_0} \sum_{j\in I} \underbrace{e_i \lambda_j a_j}_{\lambda_j (e_i a_j)} \;, \text{ where } e_i a_j = \begin{cases} 0 & \text{if } h(a_j) \neq i \\ a_j & \text{if } h(a_j) = i \end{cases}$$

$$= \sum_{i\in Q_0} \sum_{a_j : h(a_j) = i} \lambda_j a_j = \sum_{j\in J} \lambda_j a_j = a.$$

Similarly, show that $a \cdot \sum_{i\in Q_0} e_i = a$. $\square$

<u>Note</u>: The e_i form a complete family of orthogonal idempotents.

<u>Examples</u> 1.6.4:

(1) Let Q be the quiver: $1 \circlearrowleft^{\alpha}$. $|Q_0| = 1$ and paths in Q are $e_1, \alpha, \alpha^2, \alpha^3, \alpha^4, \ldots$ Thus the algebra KQ has basis $\{\alpha^j, j \in \mathbb{N}\}$ and multiplication is given as $\alpha^s \alpha^t = \alpha^{s+t}$.

$\Longrightarrow KQ$ is commutative, in fact, $KQ \cong K[x]$.

(2) Let $Q: 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \xrightarrow{\alpha_{n-1}} n$, the paths are:

Write as matrix: e^{ij} = path from j to i:

$1j$: path j to 1 only e_1
$2j$: path j to 2 e_2, α_1
$\vdots$

$$\begin{bmatrix} e_1 & 0 & 0 & & \\ \alpha_1 & e_2 & 0 & & \\ \alpha_2\alpha_1 & \alpha_2 & e_3 & & 0 \\ \alpha_3\alpha_2\alpha_1 & \alpha_3\alpha_2 & \alpha_3 & e_4 & \\ \vdots & & & \ddots & \ddots \\ \alpha_{n-1}\cdots\alpha_1 & & & \alpha_{n-1} & e_n \end{bmatrix}$$

Here KQ is $\cong$ algebra of lower triangular matrices over K $\subseteq M_n(K)$

(3) $Q: 1 \xrightarrow{a} 2 \circlearrowleft^{b}$ gives KQ with basis: $e_1, e_2, a, b, ba, b^2, b^2a, b^3, \ldots$

(4)
$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\ & & 4 \end{tikzcd}$$

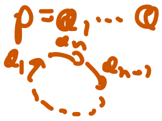
Basis:
e^{ij}: path j to i

$$\begin{bmatrix} \lambda_{e_1} & 0 & 0 & 0 \\ \lambda_\alpha & \lambda_{e_2} & 0 & 0 \\ \lambda_{\beta\alpha} & \lambda_\beta & \lambda_{e_3} & 0 \\ \lambda_{\gamma\alpha} & \lambda_\gamma & 0 & \lambda_{e_4} \end{bmatrix}$$

Note that, as a K-vector space, KQ can be written
$$KQ = KQ_0 \oplus KQ_1 \oplus KQ_2 \oplus \ldots \oplus KQ_\ell \oplus \ldots$$
where $\forall \ell \geq 0$: KQ_ℓ is the subspace of KQ of all paths of length ℓ.
Clearly, $(KQ_n)\cdot(KQ_m) \subseteq KQ_{n+m}$ $\forall n, m \geq 0$ because the product of paths of length n and m is either 0 or of length $n+m$.
Sometimes we say that this defines a grading of KQ, or KQ is a graded K-algebra.

Lemma 1.6.4 Let Q be a quiver and KQ its path algebra. Then KQ is a finite dim'l K-algebra $\Longleftrightarrow$ Q is finite and acyclic.

↳ doesn't contain oriented cycles, i.e., a path $p = a_1 \cdots a_n$ s.t. $t(a_n) = h(a_1)$



Pf: If Q is infinite, then so is the basis of KQ, so it is ∞-dim'l over K (and KQ not unitary). If Q has a cycle $w = a_n \cdots a_1$, then for each $t \geq 0$, we have a basis vector $w^t = (a_n \cdots a_1)^t$, so that KQ is again ∞-dim'l.
Conversely, if Q is finite and acyclic, it only contains finitely

many paths and so is finite dim'l over k. $\square$

Now we get to the modules of kQ:

kQ-modules are essentially the same as representations of Q:

k field $\Rightarrow$ V_i's are k-vector spaces

Def 1.6.5 A representation of Q is a tuple $V = (V_i, V_a)$ consisting of a k-module V_i for each vertex i and a k-module map $V_a : V_i \to V_j$ for each arrow $a : i \to j$ in Q. (If there is no risk of confusion, write $a : V_i \to V_j$ instead V_a).

If V is a fin. dim'l rep (all $\overset{\dim}{(V_i)} < \infty$), then its dimension vector is $\underline{\dim} V = (\dim V_i) \in \mathbb{N}^{Q_0}$.

Examples:

(1) $Q: 1 \to 2$. Have representations $V: k \xrightarrow{1} k$, $V': k \xrightarrow{0} 0$, $V'': 0 \xrightarrow{0} k$, $V''': k^2 \xrightarrow{\left[\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right]} k^3$

with dim. vectors $\underline{\dim} V = (1,1)$, $\underline{\dim} V' = (1,0)$, $\underline{\dim} V'' = (0,1)$, $\underline{\dim} V''' = (2,3)$.

(2) Q:
$$\begin{tikzcd} & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 4 \end{tikzcd}$$

lots of reps
$$\begin{tikzcd} & & 0 \\ V: 0 \arrow[r, "0"] & k \arrow[ur, "0"] \arrow[dr, "0"'] & \\ & & 0 \end{tikzcd}$$

a
$$\begin{tikzcd} & & k \\ V': k \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & k^2 \arrow[ur, "{[1\,0]}"] \arrow[dr, "{[0\,1]}"'] & \\ & & k \end{tikzcd}$$

$$\begin{tikzcd} & & 0 \\ V'': 0 \arrow[r, "0"] & k \arrow[ur, "0"] \arrow[dr, "1"'] & \\ & & k \end{tikzcd}$$

Goal of quiver representation theory is to classify those representations.

i.e. as basic finite dim'e alg. $\overset{\text{gabriel}}{\Leftrightarrow}$ kQ/I

But these are "basically the same" as finite dim'l k-algebras, so classifying modules over these algebras is the same.

Correspondence. kQ-mods $\simeq$ rep Q.

If V is a kQ-module, then $V = \bigoplus_{i \in Q_0} e_i V$

divide into paths ending at e_i

since e_i's are complete set of idemp. can write any kQ-mod in this way.

We get a representation, also denoted V, with $V_i = e_i V$, and for any arrow $a: i \to j$, the map $V_a : V_i \to V_j$ is the map given by left multiplication by $a \in \underbrace{e_j KQ e_i}_{\text{path starting at } i,\text{ ending at } j}$.

Conversely, any representation V determines a KQ-module via $V = \bigoplus_{i \in Q_0} V_i$ with the action of KQ given as:

- For $v = (v_i)_{i \in Q_0} \in V$ we have $e_i v = v_i \in V_i \subseteq V$. That is: mult. with the trivial path e_i corresponds to projection onto V_i and
- $a_1 a_2 \cdots a_n v = V_{a_1} \circ V_{a_2} \circ \dots \circ V_{a_n}(v_{t(a_n)}) \in V_{h(a_1)} \subseteq V$.

Under this correspondence:

(1) KQ-module homom. $\theta: V \to W$ correspond to tuples (θ_i) consisting of a K-module map $\theta_i : V_i \to W_i$ for each vertex i satisfying $W_a \theta_i = \theta_j V_a$ for all arrows $a: i \to j$.

$$\begin{tikzcd} V_i \arrow[r, "V_a"] \arrow[d, "\theta_i"'] \arrow[dr, phantom, "\circlearrowright"] & V_j \arrow[d, "\theta_j"] \\ W_i \arrow[r, "W_a"] & W_j \end{tikzcd}$$

$KQ = KQe_1 \oplus KQe_2$

Example: a) $Q: 1 \to 2$, $V: K \xrightarrow{1} K$ KQ-mods: $K \oplus K$

$V' : K \xrightarrow{0} 0$ $K \oplus 0$

Have a map: $V \to V'$:

$$\begin{tikzcd} K \arrow[r, "1"] \arrow[d, "\theta_1 = 1"'] & K \arrow[d, "\theta_2 = 0"] \\ K \arrow[r, "0"] & 0 \end{tikzcd}$$

We have $\theta_2 \circ id_K = 0 \circ id_K$ ✓

But no map: $V' \to V$:

$$\begin{tikzcd} K \arrow[r, "0"] \arrow[d, "\theta_1"'] & 0 \arrow[d, "\theta_2"] \\ K \arrow[r, "1"] & K \end{tikzcd}$$

because here $\theta_2 = 0$ injection $0 \to K$

$id_K \circ \theta_1 = \theta_1 \Rightarrow \theta_1 = 0$.

so $(\theta_1, \theta_2) = (0, 0)$, 0-homom.

(b) Q

$$\begin{tikzcd}
1 \arrow[dr] & & \\
& 3 & 4 \arrow[l] \\
2 \arrow[ur] & &
\end{tikzcd}$$

$V:$

$$\begin{tikzcd}
K \arrow[dr, "1"] & & \\
& K & K \arrow[l, "1"'] \\
K \arrow[ur, "1"] & &
\end{tikzcd}$$

$V' =$

$$\begin{tikzcd}
K \arrow[dr, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & & \\
& K^2 & K \arrow[l, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"'] \\
K \arrow[ur, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] & &
\end{tikzcd}$$

Then $\mathrm{Hom}_K(V', V)$ is 2-dim'l: a morphism $V' \to V$ is determ. by 5 scalars, a, b, c, d, e s.t.

$$\begin{tikzcd}
K \arrow[dr, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[ddd, bend right, "{[a]}"'] & & \\
& K^2 \arrow[ddd, "{[c\ d]}"] & K \arrow[l, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"'] \arrow[ddd, "{[e]}"] \\
K \arrow[ur, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] \arrow[ddd, bend right, "{[b]}"'] & & \\
K \arrow[dr, "1"] & & \\
& K & K \arrow[l, "1"'] \\
K \arrow[ur, "1"'] & &
\end{tikzcd}$$

commutes

get: $c + d = e$ (left)

$c = a$ (upper)

$d = b$

So choice of a, b, determines the morphism completely $\Rightarrow K^2$.

(2) Further KQ-submodules W of V correspond to tuples (W_i) where each W_i is a K-submodule of V_i, s.t. $V_a(W_i) \subseteq W_j$ for all arrows $a: i \to j$. Then W corresponds to the subrepresentation $(W_i, V_a|_{W_i}: W_i \to W_j)$ and V/W to the representation $(V_i/W_i : \overline{V}_a : V_i/W_i \to V_j/W_j)$.

(3) Direct sums of modules $V = \bigoplus_{\lambda \in \Lambda} V^\lambda$ correspond to direct sums of representations $(\bigoplus_{\lambda \in \Lambda} V_i^\lambda, \bigoplus_{\lambda \in \Lambda} V_a^\lambda)$.

Now we are approaching a theorem of P. Gabriel, that essentially says that any f. dim'l K-algebra A (with some restriction depending on $A/J(A)$) will be of the form KQ/I. ← admissible ideal