WEEK 5

We are in the middle of proving

Theorem 1.3.15 Let R be simple and artinian, M a simple (left) R-module, $D = \mathrm{End}_R(M)$ and $n = \dim_D(M) \in \mathbb{N}$.

this is well-def → every module over a div. ring is free ⇝ dim = rk of this free mod

Then: $R \cong \mathrm{End}_D M \cong M_n(D^{op})$ as rings.

Pf: Shown last time $R \cong \mathrm{End}_D M$.

Lemma 1.3.12 (4)
$$\Longrightarrow \quad \mathrm{End}_D M \cong \mathrm{End}_D(D^n) \cong M_n(\mathrm{End}_D(D)) \text{ as rings}$$

So it just remains to show that: $\mathrm{End}_D D \cong D^{op}$ (as rings).

Consider the map $\psi: D^{op} \to \mathrm{End}_D D$
$$a \mapsto f_a: D \to D : x \mapsto x \cdot a$$

ψ is inj. [$\psi(a)(x) = 0 \ \forall x \in D \Rightarrow x \cdot a = 0 \ \forall x \in D \Leftrightarrow a = 0$]

For any $g \in \mathrm{End}_D D: (x \mapsto g(x))$, then $\forall x \in D$: $g(x) = g(x \cdot 1) = x \cdot g(1)$

$\Rightarrow g(x) = \underbrace{\psi(g(1))(x)}_{x \cdot g(1)} \Longrightarrow \psi$ is surjective.

ψ is a ring hom: additive ✓

$$\psi(1_D) = (x \mapsto x \cdot 1_D = x) = \mathrm{Id}_{\mathrm{End}_D M} \quad ✓$$
$$\psi(a \cdot^{op} b)(x) = \psi(b \cdot a)(x) = x \cdot (b \cdot a) = x \cdot b \cdot a$$
$$\psi(a)(\psi(b)(x)) = \psi(a)(x \cdot b) = (x \cdot b) \cdot a = x \cdot b \cdot a \quad ✓$$

$\Rightarrow D^{op} \cong \mathrm{End}_D(D)$. □

Cor 1.3.16 For a simple ring R, TFAE:

(a) R is left artinian.

(b) R has a minimal left ideal.

(c) R is semi-simple.

(d) R is isomorphic to a matrix algebra over a division ring.

Pf: This follows from Thms 1.3.11, 1.3.15, and Ex. 1.3.5.

Theorem 1.3.17 (Artin-Wedderburn) or: "Wedderburn-Artin" or "WHAM": 1893 Molien: Thm for a. fin. dim. algs

1907: Wedderburn: f. dim. algs over arb. fields

1927: E. Artin: ACC + DCC

1939: Hopkins: ACC follows from DCC

TFAE:

(a) R is semi-simple.

(b) $R \cong M_{n_1}(D_1) \times \ldots \times M_{n_s}(D_s)$ where $s, n_1, \ldots, n_s \in \mathbb{N}$ and $D_1, \ldots, D_s$ are division rings.

Moreover, if R satisfies these conditions, then, $s, n_1, \ldots, n_s$, and $D_1, \ldots, D_s$ are uniquely determined.

Pf: (a) $\Rightarrow$ (b) By Thm 1.3.10 R is a direct sum of simple rings, and each of them is semi-simple. ($R = \bigoplus_{i=1}^{s} R_i \rightsquigarrow$ each $R_i = \bigoplus_{j=1}^{n_i} L_{ij}$ the L_{ij} are also R_i-mod. simple)

The statement follows from Cor 1.3.16.

(b) $\Rightarrow$ (a) By our example 1.3.5, matrix rings over division rings are semi-simple (and simple!) and thus their product is semi-simple.

Proof of uniqueness: Suppose that

$$M_{n_1}(D_1) \oplus \ldots \oplus M_{n_s}(D_s) \cong M_{m_1}(D_1') \oplus \ldots \oplus M_{m_r}(D_r') \text{ as rings,}$$

where $r, s, n_1, \ldots, n_s, m_1, \ldots, m_s \in \mathbb{N}_{\geq 1}$ and D_i, D_i' are division rings.

Since this is an isom. of semi-simple rings, their 2-sided ideals coincide $\overset{\text{Thm 1.3.10}}{\Longrightarrow}$ $r=s$ and (after possible renumbering) $M_{n_i}(D_i) \cong M_{m_i}(D_i')$ $\forall i \in \{1,\dots,s\}$, so we only have to verify the following:

<u>Claim</u> Let D, D' division rings, $m,n \geq 1$, such that $M_n(D) \underset{\substack{\downarrow \\ \text{assume this} \\ \text{is called } \varphi}}{\cong} M_m(D')$.
Then $m=n$ and $D \cong D'$.

Pf of claim: Set $R := M_n(D)$ and $R' = M_m(D')$. Then

$$L = \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D & 0 & \cdots & 0 \end{bmatrix}}_{n} \Bigg\} n = Re \text{ with } e = \underbrace{\begin{bmatrix} 1 & 0 & & \\ 0 & \ddots & & \\ & & & 0 \end{bmatrix}}_{\text{idempotent}} \in R.$$

L is a min. left ideal of R. Similarly,

$$L' = \begin{bmatrix} D' & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D' & 0 & \cdots & 0 \end{bmatrix} = R'f \quad \text{is min. left ideal in } R',$$

The given ring isomorphism $\varphi : R \to R'$, $R\underset{\substack{\text{idemp.}\\ \text{of } R}}{e} \mapsto R\underset{\substack{\text{idemp.}\\ \text{of } R'}}{e'}$ $\quad R \xrightarrow{\varphi} R' \xrightarrow{\cong} R'$ s.t. $e' \mapsto f$

induces (via base change in R') an isom $\psi : R \to R'$ mapping $e \mapsto f$ and L to L' and hence $D \cong eRe \mapsto fR'f \cong D'$.

Finally: $\left.\begin{array}{l} \dim_D(R) = n^2 \\ \dim_{D'}(R') = m^2 \end{array}\right\} n^2 = m^2 \Rightarrow n = m.$ $\square$

<u>Remark 1.3.18</u>

(1) By Thm 1.3.17 we also obtain the following symmetric statements: For a ring R TFAE:

(a) R is semi-simple (i.e. R is ss. left-R-module, written ${}_R R$)

(b) R is a finite direct product of simple left artinian rings.

(c) $R \cong M_{n_1}(D_1) \oplus \dots \oplus M_{n_s}(D_s)$, with parameters as in 1.3.17

(d) R_R is a semi-simple right module.

(e) R is a finite direct product of simple right artinian rings.

Similar statements for simple rings

(2) A commutative semi-simple ring is isomorphic to a finite direct product of fields.

(3) A semi-simple ring having no nonzero zero-divisors is a division ring.

§1.4 Jacobson Radical

In this section: Nakayama's lemma!

Def 1.4.1. Let M be an R-module. The Jacobson radical $J(M)$ of M is the intersection of all maximal submodules of M. proper! $(N \subsetneq M)$

(Convention: If M doesn't have any max. submodules, then $J(M)=M$).

Examples 1.4.2

(1) From comm. algebra: $\underset{R \text{ comm.}}{J(R)} = \bigcap_{m \in \operatorname{Specmax}(R)} m$

$\rightsquigarrow$ If R is local, i.e. (R, m) only has one max. ideal then $J(R) = m$ $\leftarrow$ useful for NAK

(2) Let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \subseteq M_2(k)$ upper Adler matrices

left ideals $\underset{\text{max}}{\begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}}$, $\underbrace{\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}}_{\text{not max}}$, $\underbrace{\begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}}_{\text{max}}$

$$\begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix} \cap \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} = J(R).$$

(3) If M is semi-simple, then $J(M) = 0$.

Pf: If M is simple, then 0 is the only max. submodule $\Rightarrow J(M)=0$.

If $M = \bigoplus_{i \in I} M_i$, all M_i simple, $|I| \geq 2$, then $\forall i \in I$, $N_i = \bigoplus_{j \in I \setminus \{i\}} M_j$ is a maximal submodule of M. Clearly $\bigcap_{i \in I} N_i = 0$. $\square$

(4) $J(\mathbb{Z}) = \bigcap_{p \text{ prime}} p\mathbb{Z} = 0$

(5) $J(M) = \bigcap \ker\varphi$, where the intersection runs over all $\varphi : M \to E$ with E simple.

Pf: If $0 \neq \varphi : M \to E$, where E is simple, then φ must be surjective [$\mathrm{im}(\varphi)$ is submodule of E] $\underset{\text{Hom. Thm}}{\Rightarrow}$ $E \cong M/\ker\varphi$.

$\Rightarrow \ker\varphi$ is maximal (see Rmk 1.2.2 (2)(c))

Conversely, every max. submodule $N \subset M$ is the kernel of the projection $\pi : M \to \underbrace{M/N}_{\text{simple by Rmk 1.2.2}}$.

Lemma 1.4.3 (1) If $\varphi : M_1 \to M_2$, then $\varphi(J(M_1)) \subseteq J(M_2)$.

(2) If $N \subset M$ is a submodule with $N \subset J(M)$, then $J(M/N) = J(M)/N$.

(3) $J(M/J(M)) = 0$.

(4) If $(M_i)_{i \in I}$ is a family of R-modules, then $J(\bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} J(M_i)$.

Pf: (1) For every homom. $\psi : M_2 \to E$, where E is simple, we get a homom. $\psi \circ \varphi : M_1 \to E \underset{1.4.2.(5)}{\Rightarrow} J(M_1) \subset \ker(\psi \circ \varphi)$

$M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} E$

$\Rightarrow \varphi(J(M_1)) \subset \ker(\psi)$. Set $m_1 \in J(M_1) \Rightarrow \psi(\varphi(m_1)) = 0$

$\psi(\varphi(J(M_1))) = 0 \Rightarrow \varphi(J(M_1)) \subset \ker\psi$.

Since $J(M_2) = \bigcap_{\substack{f : M_2 \to S \\ S \text{ simple}}} \ker(f) \Rightarrow \varphi(J(M_2)) \subseteq \ker(\psi)$.

(2) The maximal submodules of M/N are of the form M'/N with $N \subseteq M'$ and $M' \subset M$ maximal. Since $N \subseteq J(M)$, for any such $\overset{\text{max}}{M'}$ we have $N \subset M'$. Thus $J(M/N) = \bigcap_{\substack{M' \subset M \\ \text{max.}}} M'/N$. By def. $J(M)/N := (\bigcap_{\substack{M' \subseteq M \\ \text{max}}} M')/N$

$= \bigcap_{M' \subseteq M \text{ max}} M'/N$.

(3) Set $N = J(M)$ in (2): $J(M/J(M)) = J(M)/J(M) = 0$. □

(4) We set $M := \bigoplus_{i \in I} M_i$. Consider $\varepsilon_i : M_i \hookrightarrow M$ inclusion. Then by (1), $\underbrace{\varepsilon_i(J(M_i))}_{J(M_i)} \subseteq J(M)\ \forall i \in I \Rightarrow \bigoplus_{i \in I} J(M_i) \subseteq J(M)$.

For the other inclusion, consider $p_i : M \to M_i$ projection. Then (1) says that $p_i(J(M)) \subseteq J(M_i) \subseteq \bigoplus_{i\in I} J(M_i)$ for any $i \in I$. and hence $J(M) \subseteq \bigoplus_{i\in I} J(M_i)$. □

Cor 1.4.4. If M is an artinian R-module, then $M/J(M)$ is semi-simple.

Pf: Since M is artinian and intersections of modules forms a descending chain ($M_1 \supseteq M_1 \cap M_2 \supseteq M_1 \cap M_2 \cap M_3 \supseteq \dots$ for any submods $M_i \subseteq M, i \in I$), the chain of inters. of max. submods must become stationary, so it is enough to consider finitely many: $J(M) = \bigcap_{i=1}^{n} M_i$ for some $M_i \subseteq M$ max.. Thus have ex. seq:

$$0 \to J(M) \to M \to \bigoplus_{i=1}^{n} \underbrace{M/M_i}.$$

M_i max in $M \Rightarrow M/M_i$ is simple

From this get injective map $\varphi : M/J(M) \to \bigoplus_{i=1}^{n} M/M_i$
$m + J(M) \mapsto (m + M_i)$

$\Rightarrow \varphi(M/J(M))$ $\cong M/J(M)$ since φ injective! is isom. to a submodule of a s.-s. module and thus itself semi-simple. □

Def 1.4.5

(1) An element $a \in R$ is called nilpotent if $\exists n \geq 1$ s.t. $a^n = 0$.

(2) A subset $I \subset R$ is called nil if every $a \in I$ is nilpotent.

(3) A left (resp. right, resp. two-sided) ideal I is called nilpotent if there is an $n \geq 1$ such that $I^n = 0$.

Examples: • By def. every nilpotent ideal is nil.

• In $R = \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{bmatrix}$ = ring of upper triangular matrices, any matrix $\begin{bmatrix} 0 & * & * \\ & \ddots & * \\ 0 & & 0 \end{bmatrix}$ is nilpotent. Hence $I = \begin{bmatrix} 0 & * & * \\ & \ddots & * \\ 0 & & 0 \end{bmatrix}$ is nilpotent 2-sided ideal

Lemma 1.4.6 (1) $J(R) = \bigcap_{E \text{ simple}} \underbrace{\mathrm{ann}_R(E)}_{\{r \in R : r \cdot e = 0 \ \forall e \in E\}} \subset R$ is a 2-sided ideal

(2) If $I \subseteq R$ is a nil left ideal, then $I \subset J(R)$.

Pf (1) Any annihilator $\operatorname{ann}_R(M)$ of a left R-module is a 2-sided ideal: $\operatorname{ann}_R(M) = \{x \in R : x \cdot m = 0 \ \forall m \in M\}$

Let $r \in R$, $x \in \operatorname{ann}_R(M)$: $(r \cdot x) \cdot m = r(xm) = r \cdot 0 = 0 \Rightarrow rx \in \operatorname{ann}_R(M)$

$(x \cdot r) \cdot m = x \cdot \underbrace{(rm)}_{\in M} = 0 \Rightarrow xr \in \operatorname{ann}_R(M)$.

Thus enough to show $J(R) = \bigcap_{E \text{ simple}} \operatorname{ann}_R(E)$.

isom. thm: $Rx \cong R/\operatorname{ann}_R x$.

$0 \to \operatorname{ann}_R x \to R \xrightarrow{\cdot x} R \to 0$

If E is simple, then write $E = Rx$ for any $x \neq 0 \in E \Rightarrow E \cong R/\operatorname{ann}_R(x)$.

$\Rightarrow \operatorname{ann}_R(x)$ is a maximal left ideal of R.

Conversely, if $I \subseteq R$ is a maximal left ideal, then R/I is a simple R-module and $I = \operatorname{ann}_R(1+I)$.

$x \in \operatorname{ann}_R(1+I) \Leftrightarrow x \cdot (1+I) = 0 + I \Leftrightarrow x + I = 0 + I \Leftrightarrow x \in I$.

Thus we have $J(R) = \bigcap_{E \text{ simple}} \left(\bigcap_{x \in E} \operatorname{ann}_R(x) \right) = \bigcap_{E \text{ simple}} \operatorname{ann}_R(E)$.

(2) Let $a \in I$. By (1) it suffices to show that $aE = 0$ $\forall E$ simple. Let E be a simple R-module and assume on the contrary that $\exists x \neq 0 \in E$ with $ax \neq 0$. Then $R(ax) = E$, and thus $\exists b \in R$ s.t. $b(ax) = x$. Then $(ba)^2 x = ba(bax) = bax = x \rightsquigarrow (ba)^n x = x \ \forall n$. But $a \in I$ implies $ba \in I$ and ba nilpotent, i.e. $\exists m \geq 1$ s.t. $(ba)^m = 0$ $\lightning$. $\square$

Now we will approach the noncomm. Nakayama lemma, we need some notation first:

Def 1.4.7 Let M be an R-module. A submodule $M' \subseteq M$ is called superfluous (in M) if for all submodules $N \subseteq M$ the equation $N + M' = M$ implies that $M = N$.

Lemma 1.4.8 Let M be an R-module.

(1) TFAE: (a) M is finitely generated.

(b) $M/J(M)$ is f.g. and $J(M)$ is superfluous.

(2) $J(R)M \subset J(M)$.

(3) NAKAYAMA'S LEMMA: If M is finitely generated, then $J(R)M$ is superfluous.

Note: Nakayama's lemma is usually used in the following way: Let M be a f.g. module with $J(R)M = M$. Since (*) $J(R)M$ is superfluous by Nakayama's lemma and $M = J(R)M + 0$, it follows that $M = 0$.

Further use: let M f.g. $N \subseteq M$ submodule, and assume that: $M = J(R)M + N$. Then $M = N$

(apply (*) to M/N: $M/N = J(R)\cdot M/N + 0 \Longrightarrow M/N = 0$)

2nd isom. $(I+J)/I \cong J/I \cap J$