Recall Thm 1.3.7.:

(1) Let $R = \bigoplus_{i=1}^{s} L_i$, L_i minimal left ideals.
Then $\exists e_1, \dots, e_s \in R$ s.t:

(a) $e_1, \dots, e_s$ are nonzero idempotents,

(b) For any $i, j \in \{1, \dots, s\}$, $i \neq j$, we have $e_i e_j = 0$.

(c) $1 = e_1 + \dots + e_s$

(d) Any e_i, $i \in \{1, \dots, s\}$, cannot be written as $e_i = e_i' + e_i''$ with e_i', e_i'' nonzero and $e_i' e_i'' = 0$.

(2) Let R be s.-s. and let $e_1, \dots, e_s$ be elements satisfying (1)(a)-(d). Then the ideals $Re_1, \dots, Re_s$ are minimal and $R = \bigoplus_{i=1}^{s} Re_i$.

Def A family $(e_1, \dots, e_s)$ satisfying properties (a)-(c) is called a complete system of orthogonal idempotents. If, in addition, (d) holds, then it is a primitive system of idempotents.

Lemma 1.3.8 Let R be semi-simple, $L \subseteq R$ a minimal left ideal, and M a simple R-module. Then

$$LM = \begin{cases} 0 & \text{if } L \not\cong M \\ M & \text{if } L \cong M \end{cases}$$

Pf: Suppose $LM \neq 0$. Then $\exists x \in L$, $m \in M$, s.t. $0 \neq xm \in Lm \subseteq M$.
Since M is simple, we must have: $Lm = LM = M$. Then $f: L \xrightarrow{\cdot m} M : l \mapsto lm$ is an R-module epimorphism with $\ker(f) \subsetneq L$. Since L is minimal $\Rightarrow \ker(f) = 0 \Rightarrow f$ is isomorphism.
If $f: L \to M$ is an isom., then by Thm 1.3.6 (R s.s., left id: $L = Re$), there is an

idempotent $e \in R$ with $Re = L$. Set $m_0 := f(e)$. ② (oth: $M=0$)
Since $f(re) = r \cdot f(e) = r \cdot m_0 \ \forall r \in R$, we must have $m_0 \neq 0$.
Then $m_0 = f(e) = f(e^2) = e \cdot f(e) = e m_0$ shows that $LM \neq 0$.
Since M is simple, this also shows that $LM = M$. □

Lemma 1.3.9 Let R be semi-simple.
(1) Let $L \subseteq R$ be a minimal left ideal. The sum of all ideals isomorphic to L is a minimal two-sided ideal.
(2) Let $I \subseteq R$ be a two-sided ideal containing a minimal left ideal L. Then I contains all left ideals isomorphic to L.

Pf (1) (a) Set $A = \sum J$, where the sum ranges over all left ideals $J \subset R$, which are isomorphic to L. Then clearly A is a left ideal. We have to show that it is also a right ideal. By Thm 1.3.3 we may write $R = \bigoplus_{i=1}^{s} L_i$, where L_i is a min. left ideal.
Then
$$AR = \sum_{J \leq R, J \cong L} JR = \sum_J \sum_{i=1}^{s} JL_i.$$
By Lemma 1.3.8: $\forall i \in \{1, \dots, s\}$:
$$JL_i = \begin{cases} 0 & \text{if } J \ncong L_i \\ L_i & \text{if } J \cong L_i \end{cases}$$
and thus $AR \subset A$. Before showing that L is minimal, show (2):
(2) Let $J \subseteq R$ be a left ideal with $J \cong L$. Then $1.3.8 \Rightarrow J = LJ \subseteq I$. $L \subseteq I$
Now (1)(b): Let $A = \sum_{J \leq R, J \cong L} J$, and let $0 \neq A_1 \subseteq A$ be a two-sided ideal of R. We show that $A_1 = A$, which implies that $A = A_1$ is a minimal 2-sided ideal.
Let therefore $L_1 \subseteq A_1$ be a minimal left ideal with $L_1 \subseteq A_1$ ($L_1 \neq 0$). If $L_1 \ncong L$ then $L_1 J = 0 \ \forall J \cong L$. Then $L_1 A = 0$, and thus also $L_1 \underbrace{L_1}_{L_1 \subseteq A_1 \subseteq A} = 0$.

But this is ↯ because L_1 contains a nonzero idempotent. $\left(\begin{smallmatrix} R \text{ s.s.} \\ \Rightarrow L_1 \cong Re \end{smallmatrix}\right)$ ③
$\Rightarrow L_1 \cong L$, and thus (2) implies $A \subset A_1$. $\square$

Thm 1.3.10

Let $R = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{r_i} L_{ij}$ be a semi-simple ring, where for all $i \in \{1, \dots, s\}$ and all $j \in \{1, \dots, r_i\}$, L_{ij} is a minimal left ideal such that $L_{i1} \cong L_{ij}$ and $L_{11}, \dots, L_{s1}$ are pairwise non-isomorphic. For $i \in \{1, \dots, s\}$, set $R_i := \bigoplus_{j=1}^{r_i} L_{ij}$.

(1) For every $i \in \{1, \dots, s\}$ $R_i \subset R$ is a minimal two-sided ideal.
(2) For all $i, j \in \{1, \dots, s\}$, $i \neq j$, we have $R_i R_j = 0$.
(3) Every two-sided ideal I of R has the form $I = \bigoplus_{j \in J} R_j$ for some $J \subset \{1, \dots, s\}$.
(4) $R = R_1 \oplus \dots \oplus R_s$ is a direct sum of simple rings $R_1, \dots, R_s$.
(5) There are elements $e_1, \dots, e_s \in R$ s.t.:
   (a) $e_1, \dots, e_s$ are nonzero central idempotents ← i.e. $e_i \in Z(R)$
   (b) $\forall i, j \in \{1, \dots, s\}$ with $i \neq j$, we have $e_i e_j = 0$.
   (c) $1 = e_1 + \dots + e_s$.
   (d) $\forall i \in \{1, \dots, s\}$, e_i cannot be written in the form $e_i = e_i' + e_i''$ with nonzero central idempotents e_i', e_i'' with $e_i' e_i'' = 0$.

⇝ $\{e_1, \dots, e_s\}$ is called a system of central primitive idempotents.

Ex: • $R = M_2(k)$, k field $L_1 = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}$, $L_2 = \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$ $L_1 \xrightarrow{\cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} L_2$ isom: $\neq 0$ Schur

$L_1 \cong L_2$

$\Rightarrow s = 1$, $r_1 = 2$ $R_1 = L_{11} \oplus L_{12} = R$ is a simple ring (ex. 1.3.5)

• $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ not semi-simple (!)

• $R = M_2(k) \times M_2(k)$ $\begin{pmatrix} k & k & 0 & 0 \\ k & k & 0 & 0 \\ 0 & 0 & k & k \\ 0 & 0 & k & k \end{pmatrix}$ 4 idempotents: $L_i = Re_i$ $L_1 \cong L_2 \rightsquigarrow L_{11}, L_{12}$
$L_3 \cong L_4 \rightsquigarrow L_{21}, L_{22}$

Get $R = \underbrace{R_1}_{\cong M_2(k)} \oplus \underbrace{R_2}_{\cong M_2(k)}$

Pf: (1) This follows from Lemma 1.3.9.

(2) Follows from Lemma 1.3.8.

(3) Let $I \subset R$ be a two-sided ideal. Then $I = R \cap I = \bigoplus_{i=1}^{s} (R_i \cap I)$

Since R_i are minimal two-sided ideals, we have $R_i \cap I = \begin{cases} 0 & \text{or} \\ R_i \end{cases}$. $\Rightarrow I = \bigoplus_{\substack{i=1 \\ \text{s.t. } R_i \cap I \neq 0}}^{s} R_i$.

(4) Let $i \in \{1, \dots, s\}$. We already know $R = R_1 \oplus \dots \oplus R_s$. Have to show that R_i is a simple ring. ($\leftarrow$ do not know that R_i has unit!!)

Let $B_i \subset R_i$ be a two-sided ideal $\rightsquigarrow$ show that B_i is 2-sided ideal in R, then (3) $\Rightarrow B_i = \begin{cases} R_i \\ 0 \end{cases}$

$\Rightarrow R_i$ simple.

B_i ideal in $R_i \Rightarrow B_i$ closed under $+$.

Let $b \in B_i$ and $r \in R$. Then $r = x_1 + \dots + x_s$ where $x_i \in R_i$ and $r \cdot b = \sum_{j=1}^{s} x_j b = x_i b \in B_i$, since $B_i \subset R_i$ is an ideal.

Since B_i is 2-sided ideal in R_i, also get $br \in B_i \Rightarrow B_i$ 2-sided ideal in R.

(5) The proof runs along the same lines as Thm 1.3.7.

We show that the $e_1, \dots, e_s$ are central elements: $\rightsquigarrow x e_i = e_i x \ \forall x \in R$

Let $1 = e_1 + \dots + e_s$ with $e_i \in R_i \ \forall i \in \{1, \dots, s\}$.

Let $x \in R$: $x \cdot 1 = x \sum_{i=1}^{s} e_i = 1 \cdot x = \sum_{i=1}^{s} e_i x$.

direct sum!

Since $R_i \subset R$ are two-sided ideals and $R = \bigoplus_{i=1}^{s} R_i$

$\Rightarrow x e_i = e_i x \ \forall i \in \{1, \dots, s\}$

In part, $e_i \in R$ is the identity element of R_i and R_i is a ring $\forall i \in \{1, \dots, s\}$. $\square$

Now we go on to study the structure of simple rings:

Thm 1.3.11 For a simple ring R, TFAE:
(a) R is left artinian.
(b) R has a minimal left ideal.
(c) R is semi-simple.

$\Leftrightarrow$ every nonempty set of left ideals has min. element

Proof (a) $\Rightarrow$ (b): Since R is left artinian the set of nonzero left ideals has a minimal element.
(b) $\Rightarrow$ (c): Let $L \subset R$ be a minimal left ideal. Then $\sum_{r \in R} Lr$ is also a two-sided ideal. $\left(\begin{array}{l}\text{left: def.}\\ \text{right: } x = \sum_{r\in R} \ell_r r \in \sum Lr \quad x \cdot \overset{R}{\underset{\in}{s}} = \sum \ell_r \underbrace{rs}_{r' \in R} \in \sum_{r'=rs \in R} Lr' \subset \sum_{r \in R} Lr\end{array}\right)$

Moreover $0 \neq L \cdot 1 \subset \sum_{r \in R} Lr$ implies that $R = \sum_{r \in R} Lr$ (because R simple!)

Thus R is the sum of simple left modules $(Lr)_{r \in R}$ and hence semi-simple.
(c) $\Rightarrow$ (a): This was shown in Thm 1.3.3. $\square$

Next goal: understand morphisms betw. modules: Write morphisms between $\oplus$'s as "matrices"

Lemma 1.3.12:
Let M, N be R-modules having a direct sum decomposition $M = \bigoplus_{j=1}^{r} M_j$, $N = \bigoplus_{i=1}^{s} N_i$. For $j \in \{1, \dots, r\}$, let $\varepsilon_j : M_j \to M$ be the embedd.

$\varepsilon_j(m_j) = (0, \dots, \underset{j\text{-th}}{m_j}, 0, \dots, 0)$

and for $i \in \{1, \dots, s\}$ let $p_i : N \to N_i$ be the projection. $p_i(n_1, \dots, n_i, \dots, n_s) = n_i$

(1) For $i \in \{1, \dots, s\}$, $j \in \{1, \dots, r\}$ let $\Phi_{ij} \in \mathrm{Hom}_R(M_j, N_i)$. Then make a map $\Phi : M \to N$ from these:

$$\Phi = (\Phi_{ij})_{\substack{i=1,\dots,s\\ j=1,\dots,r}} : M_1 \oplus \dots \oplus M_r \longrightarrow N_1 \oplus \dots \oplus N_s$$

$$\underset{\text{column}}{(m_1 + \dots + m_r)} \longmapsto \underbrace{\begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1r} \\ \vdots & & \\ \Phi_{s1} & \cdots & \Phi_{sr} \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix}}_{[(\Phi_{11}(m_1) + \dots + \Phi_{1r}(m_r)) + \dots + (\Phi_{s1}(m_1) + \dots + \Phi_{sr}(m_r))]}$$

(2) If $\Phi \in \mathrm{Hom}_R(M, N)$ is any R-homom., and $\Phi_{ij} := p_i \circ \Phi \circ \varepsilon_j \in \mathrm{Hom}(M_j, N_i)$

then $\Phi = (\Phi_{ij})$.

(3) If $\Phi = (\Phi_{ij})$ and $\psi = (\psi_{ij})$, then $\Phi + \psi = (\Phi_{ij} + \psi_{ij})$

(4) $\mathrm{End}_R(M^n) \cong M_n(\mathrm{End}_R(M))$ (as rings).

Pf: Exercise (obvious!) $\square$

Lemma 1.3.13 Let M be an R-module and let $S = \mathrm{End}_R M$.

(1) The map $\alpha : S \times M \to M$

$$(\varphi, m) \mapsto \varphi \cdot m := \varphi(m)$$

defines an S-module structure on M.

(2) If M is s.-s., $m \in M$, and $\varphi \in \mathrm{End}_S M$, then there is an $a \in R$ s.t. $\varphi(m) = am$.

Pf (1) Show:
- $\alpha(\varphi + \varphi', m) = \varphi \cdot m + \varphi' \cdot m \to \alpha(\varphi + \varphi', m) = (\varphi + \varphi')(m) = \varphi(m) + \varphi'(m) = \varphi \cdot m + \varphi' \cdot m$
- $\alpha(\varphi, m + m') = \varphi \cdot m + \varphi \cdot m' \to \mathrm{LHS} = \varphi(m+m') = \varphi(m) + \varphi(m') = \varphi \cdot m + \varphi m'$
- $\alpha(\varphi, \alpha(\varphi', m)) = \alpha(\varphi\varphi', m) \to \mathrm{LHS}: \alpha(\varphi, \varphi'(m)) = \varphi(\varphi'(m)) = (\varphi\varphi')(m) = \alpha(\varphi\varphi', m)$.

(!)

(2) Since M is s.-s., there is a $W \subset M$ such that $M = Rm \oplus W$. Thm 1.2.4

Let $p : M \to M$ be the projection onto Rm: $p(rm + w) = rm$.

Then clearly $p \in S = \mathrm{Hom}_R(M, M)$. If $\varphi \in \mathrm{End}_S(M)$, then

$\underset{(m,0) \in Rm \oplus W}{\varphi(m)} = \varphi(\underset{p \cdot m}{p(m)}) = \underbrace{p\varphi(m)}_{\text{Because ring map: } \varphi(p \cdot m) = p \cdot \varphi(m)} \in Rm$. Thus $\exists a \in R$ s.t.

$$\varphi(m) = a \cdot m. \qquad \square$$

For every $a \in R$,

$\ell_a : M \to M$, $m \mapsto am$ is an element of $\mathrm{End}_S(M)$ and the map

$$R \to \mathrm{End}_S M$$
$$a \mapsto \ell_a$$

is a ring homom.

The next result states that in a suitable topology on $\mathrm{End}_S M$

the set $\{\ell_a : a \in R\}$ is dense in $\mathrm{End}_S M$:

<u>Theorem</u> 1.3.14 (JACOBSON'S DENSITY THEOREM)
Let M be a s.-s. module, $S = \mathrm{End}_R M$, and $f \in \mathrm{End}_S M$. If $n \in \mathbb{N}$ and $m_1, \dots, m_n \in M$, then there is an $a \in R$ s.t. $f(m_i) = a m_i \ \forall i \in \{1, \dots, n\}$

<u>Pf</u>: Let $n \in \mathbb{N}$ and $m_1, \dots, m_n \in M$. We set $S' = \mathrm{End}_R(M^n)$ and define

$$f^{(n)} : M^n \to M^n$$
$$x_1 + \dots + x_n \mapsto f(x_1) + \dots + f(x_n)$$

Δ-map " $\begin{bmatrix} f & & 0 \\ & \ddots & \\ 0 & & f \end{bmatrix} \circ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ "

<u>Claim</u>: $f^n \in \mathrm{End}_{S'}(M^n)$.

If Claim holds, then Lemma 1.3.13 implies that there is an $a \in R$ s.t. $f^{(n)}(m_1 + \dots + m_n) = a(m_1 + \dots + m_n) = a m_1 + \dots + a m_n$
$\Rightarrow f^{(n)}(m_i) = f(m_i) = a m_i$. $\forall i \in \{1, \dots, n\}$.

<u>Proof of Claim</u>: Let $\Phi \in S' = \mathrm{End}_R(M^n)$. By Lemma 1.3.12, we can write $\Phi = \Phi_{ij}$ with $\Phi_{ij} \in \mathrm{Hom}_R(M_j, M_i)$ $\forall i, j \in \{1, \dots, n\}$.
We have to show that $f^{(n)} : M^n \to M^n$ is S' linear:
Clearly $f^{(n)}$ is additive, for multiplication in S' compute

$$\begin{aligned}
f^{(n)}(\Phi \cdot (x_1 + \dots + x_n)) &= f^n(\Phi \circ (x_1 + \dots + x_n)) \\
&= f^n(\Phi_{11}(x_1) + \dots + \Phi_{1n}(x_n) + \\
&\quad + \Phi_{21}(x_1) + \dots + \Phi_{2n}(x_n) + \dots \\
&\quad + \Phi_{n1}(x_1) + \dots + \Phi_{nn}(x_n)) \\
&= f(\Phi_{11}(x_1)) + \dots + f(\Phi_{1n}(x_n)) + \dots + f(\Phi_{nn}(x_n)) \\
&\overset{f \text{ is ring element in } S}{=} \Phi_{11}(f(x_1)) + \dots + \Phi_{1n}(f(x_n)) + \dots + \Phi_{nn}(f(x_n)) \\
&= \Phi(f(x_1) + \dots + f(x_n)) \\
&= \Phi(f^{(n)}(x_1 + \dots + x_n)) = \Phi \cdot f^{(n)}(x_1 + \dots + x_n).
\end{aligned}$$

✓claim $\square$

Now we approach the Wedderburn–Artin theorem (structure of s.-s. rings : $R \cong \bigoplus_{i=1}^{n} M_{n_i}(D_i)$) $\to$ first look at the case: R simple.

Let M be a right R-module. Recall: $(R^{op}, +, \cdot^{op})$ is the ring R with $a \cdot^{op} b = \underbrace{b \cdot a}_{\text{in } R}$ $\forall a,b \in R$

Define a map: $R^{op} \times M \to M$
$$(a, m) \mapsto a \cdot^{op} m = m \cdot a \quad \forall a \in R, \forall m \in M.$$
This makes M a left R^{op}-module: $\forall a,b \in R, \forall m \in M$

- associative: $(a \cdot^{op} b) \cdot^{op} m = m(a \cdot^{op} b) = m \cdot b \cdot a = m \cdot (ba)$
  $a \cdot^{op} (b \cdot^{op} m) = a \cdot^{op} (m \cdot b) = (m \cdot b) \cdot a = m \cdot (ba)$
- additive; dem: $(a+b) \cdot^{op} m = m(a+b) = ma + mb = a \cdot^{op} m + b \cdot^{op} m$
  $a \cdot^{op} (m+m') = (m+m') \cdot a = ma + m'a = a \cdot^{op} m + a \cdot^{op} m'$.

<u>Theorem 1.3.15</u> Let R be simple and artinian, M a simple (left) R-module, $D = \mathrm{End}_R(M)$ and $n = \dim_D(M) \in \mathbb{N}$.

this is well-def $\to$ every module over a div. ring is free $\rightsquigarrow$ dim = rk of this free mod

Then: $R \cong \mathrm{End}_D M \cong M_n(D^{op})$ as rings.

<u>Proof</u>: Define $\Phi : R \to \mathrm{End}_D M$
$$a \mapsto l_a : M \to M, \quad x \mapsto a \cdot x$$
this is a ring hom (!) $\Phi(a+b)(x) = (a+b)(x) = ax + bx = \Phi(a)(x) + \Phi(b)(x)$.
$\Phi(ab)(x) = (ab)x = abx$
$\Phi(a)(\Phi(b)(x)) = \Phi(a)(bx) = abx$
$\Phi(1)(x) = x = \mathrm{Id}_{\mathrm{End}_D M}$ ✓

let $a \in \ker \Phi$, $\forall b \in R$ then
$\Phi(ba) = bax = b0 = 0 \ \forall x \in M$
$\Phi(ab) = a \cdot (bx) = a \cdot \tilde{x} = 0$ ✓

Since $\ker \Phi = \{a \in R : ax = 0 \ \forall x \in M\}$ is a two-sided ideal in R and $1 \notin \ker \Phi$, it follows (from R simple), that $\ker \Phi = 0$.

$\Phi(1)(x) = \mathrm{Id}_{\mathrm{End}_D M} \neq 0$ since $M \neq 0$ M simple!

$\Rightarrow \Phi$ is monomorphism.

In order to show surjectivity, we want to use Jacobson density thm, but for this we need finitely many elements, therefore

<u>Claim</u>: $\dim_D(M) < \infty$.

<u>Pf of Claim</u>: Let $s \in \mathbb{N}$, $\{m_1, \dots, m_s\} \subset M$ be D-linearly independent and for $i \in \{1, \dots, s\}$ define

$$A_i = \{a \in R : a m_j = 0 \quad \forall j \in \{1, \dots, i\}\} \subset R$$

Clearly, $A_s \subset R$ is a left ideal. Since $m_1, \dots, m_s$ are D-linearly independent, there is, for all $y_1, \dots, y_s \in M$, some $f \in \mathrm{End}_D M$ with $f(m_i) = y_i \quad \forall i \in \{1, \dots, s\}$ [f is defined by values of a basis $\rightsquigarrow$ part of basis also gives map]

in part: $f(m_i) = 0 \quad \forall i = 1, \dots, s$

By Thm 1.3.14, $\exists a \in R$ with $f(m_i) = a m_i \overset{=0}{} \quad \forall i \in \{1, \dots, s\}$.

$$\Rightarrow A_1 \supsetneq A_2 \supsetneq \dots \supsetneq A_s.$$

$A_1 = \{a \in R : a m_1 = 0\}$ choose $y_1 = 0$, others don't care

$A_2 = \{a \in R : a m_1 = a m_2 = 0\} \rightarrow y_1 = 0, y_2 = 0$

If $a \in A_2 \Rightarrow a m_1 = 0 \Rightarrow a \in M_2$

If we choose $y_1 = 0, y_2 \neq 0 \rightsquigarrow$ get a with $a m_1 = 0$, $a m_2 \neq 0 \Rightarrow a \in A_1 \setminus A_2$ etc.

If there was an infinite set of D-linearly independent elements, then we would have an infinite descending chain of left ideals. $\lightning$ to R artinian. $\square$ Claim

Now let $\{m_1, \dots, m_n\}$ be a D-basis of M. Again, by Thm 1.3.14 for every $f \in \mathrm{End}_D(M)$, there is an $a \in R$ with $f(m_i) = a m_i \quad \forall i = 1, \dots, n$

$\Rightarrow f(m) = a m = \ell_a(m) \quad \forall m \in M$. Hence Φ is surjective.

$\Rightarrow \Phi$ is a ring isomorphism.

Since M is a free D-module of rk n: $M \cong D^n$

$$\overset{\text{Lemma 1.3.12 (4)}}{\Longrightarrow} \quad \mathrm{End}_D M \cong \mathrm{End}_D(D^n) \cong M_n(\mathrm{End}_D(D)) \text{ as rings.}$$

So it just remains to show that: $\mathrm{End}_D D \cong D^{op}$ (as rings).

Next week!