WEEK 15

Recall: R is (right) Goldie ring if R_R has finite uniform dimension and satisfies ACC on right annihilators. ($I = r\text{-ann}(S) \Leftrightarrow I = \{r \in R : sr = 0 \ \forall s \in S\}$)

- R has classical quot. ring Q (Q is quot w.r.t. $X = \{\text{reg. elts}\}$ $\Rightarrow \forall x \in X : x^{-1} \in Q$ • $q \in Q : a x^{-1} = q$)
- Prop 3.3.3: If R has class. quot Q $\Rightarrow$ R r. Goldie (R semiprime $\Rightarrow$ Q ss)

Lemma 3.3.7 Let R be a semiprime right Goldie ring and $x \in R$
TFAE: (a) x is regular.
(b) $r\text{-ann}_R(x) = 0$.
(c) $xR \subset R_R$ is an essential right ideal.

Pf: (a) $\Rightarrow$ (b): By def of regular.
(b) $\Rightarrow$ (c): Since $r\text{-ann}(x) = 0 \Rightarrow xR \cong R_R$. Thus $\text{udim}(xR) = \text{udim}(R) < \infty \Rightarrow xR \subseteq_e R_R$ is ess. by Cor 2.3.10 (2) $\Rightarrow N \subset_e M$. ($N \subset M$, $\text{udim}(N) = \text{udim}(M)$)
(c) $\Rightarrow$ (a): By Prop 3.3.6. (2), the left annihilator of an ess. right ideal is $0 \Rightarrow l\text{-ann}(xR) = 0$ and so $l\text{-ann}(x) = 0$.
We need to show that $I := r\text{-ann}(x) = 0$.
Therefore, choose a right ideal J, which is maximal s.t. $I \cap J = 0$.
Then $I \oplus J \subseteq_e R$ by Lemma 2.3.4. (6).
Claim: $xJ \subseteq_e xR$.
Pf of Claim: Choose $xr \overset{\neq 0}{} \in xR$ and set $K = \{k \in R : rk \in (I+J)\}$. $= r^{-1}(I+J)$
By Lemma 2.3.4. (3) we get that $K \subseteq_e R_R$. Now Prop 3.3.6. (2) implies that $l\text{-ann}(K) = 0$ and hence $xrK \neq 0$. However,

$$xrK \subset x(I+J) = xJ$$

whence $xR \cap xJ \neq 0$. Thus $xJ \subseteq_e xR$.
Since R has finite uniform dimension. Cor 2.3.10 implies that $\text{udim}(xJ) = \text{udim}(xR) = \text{udim}(R)$. Since $J \cap r\text{-ann}(x) = 0$, we have $J \cong xJ$ and hence $\text{udim}(J) = \text{udim}(R)$. Again by Cor 2.3.10.

$\Rightarrow J \leq_e R_R$ is essential. Thus $I = 0$ and x is regular. $\square$

<u>Lemma 3.3.8</u> Let R be a semiprime right Goldie ring. Then R satisfies the DCC for right annihilators.

<u>Pf</u>: By Prop. 3.3.6., the left annihilators of essential right ideals are 0. Let $I_1 \supset I_2 \supset \ldots$ be a descending chain of left annihilators and suppose that $I_j := r\text{-ann}(A_j)$ for each $j \in \mathbb{N}$.
Then: $\operatorname{udim}(I_1) \geq \operatorname{udim}(I_2) \geq \ldots$
Hence $\exists m \in \mathbb{N}$ such that $\operatorname{udim}(I_m) = \operatorname{udim}(I_n)$ $\forall n \geq m$.
Then Cor 2.3.10 $\Rightarrow I_n \leq_e I_m$ $\forall n \geq m$ and we show that $I_n = I_m$: let therefore $n \geq m$ be given. Choose some $x \in I_m$ and set

$$J = \{r \in R : xr \in I_n\}.$$

Then $J \leq_e R_R$ is essential and $A_n x J = 0$. Since $\ell\text{-ann}(J) = 0$, it follows that $A_n x = 0$ and hence $x \in r\text{-ann}(A_n) = I_n$. $\square$

<u>Prop 3.3.9</u> (Goldie's Regular element lemma): Let R be a semiprime Goldie ring, and let $I \subset R$ be a right ideal. Then I is an essential right ideal $\Leftrightarrow$ I contains a regular element.

<u>Pf</u>: <u>$\Leftarrow$</u>: Let $x \in I$ be a regular element. By Lemma 3.3.7.
$xR \leq_e R_R$ is essential and hence $I \leq_e R_R$ is essential.
<u>$\Rightarrow$</u>: Conversely, suppose that $I \leq_e R_R$ is essential. By Lemma 3.3.8, $\exists x \in I$ s.t. $A := r\text{-ann}(x)$ is minimal among all right annihilators of elements from I.

<u>Claim</u>: $xR \leq_e I$ is essential.

Suppose the Claim holds true. Since $I \leq_e R$ is essential, it follows that $xR \leq_e R_R$ is essential, $\Rightarrow$ x is regular by Lemma

3.3.7.

Only remains to prove the claim: Let $B \subset I$ be a right ideal with $B \cap xR = 0$. We have to show that $B = 0$.
For every $b \in B$ we have $bR = xR = 0 \Rightarrow$

$$r\text{-ann}(b+x) = r\text{ann}(b) \cap r\text{-ann}(x) \subset A.$$

Since $x+b \in I$, the minimality of A implies that

$$A = r\text{-ann}(b+x) = r\text{-ann}(b) \cap r\text{-ann}(x) \subset r\text{-ann}(b)$$

and hence $bA = 0$. Thus we obtain $BA = 0$.
Since $(B \cap A)^2 \subset BA = 0$ and R is semiprime, it follows that $B \cap A = 0$.
Similarly $(RB \cap A)^2 \subset RBA = 0$ implies that $RB \cap A = 0$.
Note that for the 2-sided ideal RB we have $xRB \subset RB$, whereas $xB \cap B \subset xR \cap B = 0$. Since $RB \cap A = 0$, left multiplication by x from $RB \xrightarrow{x} RB$ is a monomorphism $\Rightarrow xRB \cong RB$.
Since $(RB)_R$ has finite uniform dim and $\text{udim}(xRB) = \text{udim}(RB)$, Cor 2.3.10 (2) implies that $xRB \subset_e RB$ is essential.
However, since $B \cap xRB \subset B \cap xR = 0$, it follows that $B = 0$. □

<u>Cor 3.3.10</u> Let R be a prime right Goldie ring. Then every nonzero ideal of R contains a regular element.

<u>Pf</u> By Lemma 2.3.3 every nonzero ideal of R is essential and thus the assertion follows from Prop 3.3.9. □

<u>Thm 3.3.11 (Goldie's Theorem)</u> TFAE:
(a) R has a semisimple classical right quotient ring.
(b) R is a semiprime right Goldie ring.
In particular, every semiprime right noetherian ring has a semisimple right quotient ring.

<u>Pf</u>: Since every right noetherian ring has finite uniform dim.,

it is right Goldie. Thus the "in particular" statement follows.

(a) $\Rightarrow$ (b) This follows from Prop 3.3.3.

(b) $\Rightarrow$ (a) First show that the set of regular elements of R satisfy the right Ore condition. $\forall x \in X, b \in R : bX \cap xR \neq \emptyset$ $(\Leftrightarrow \exists z \in X, c \in R$ s.t. $bz = xc)$

Because then, by Thm 3.1.5. Ore-Asano R has a classical right quotient ring Q.

(a) $\Rightarrow$ (c)

Let now $a, x \in R$, and suppose that x is regular. Then, by Lemma 3.3.7 $xR \subseteq_e R_R$ is essential, and hence the right ideal $J = \{r \in R : ar \in xR\} \subseteq_e R_R$ is essential. By Prop 3.3.9., there is a regular element $y \in J$ and hence $ay = xb$ for some $b \in R$. Thus the right Ore condition holds.

It remains to show that Q is semisimple. By Lemma 2.3.4. (7) it suffices to show that Q_Q has no proper essential right ideals: let therefore $I \subseteq_e Q$ be an essential right ideal.

Claim: $I \cap R \subseteq_e R$ is essential.

If the Claim holds, then Prop 3.3.9 $\Rightarrow I \cap R$ contains a regular element y. Since y is invertible in Q, it follows that $I = Q$.

Pf of Claim: Consider a nonzero $b \in R$. We have to show that $bR \cap (I \cap R) \neq 0$. Clearly $\exists q \overset{\neq 0}{\in} Q$ s.t. $0 \neq bq \in I$. We set $q = ax^{-1}$ with $a, x \in R$ and x regular. Then $ba = bqx \neq 0$ (x regular!) is contained in $bR \cap I$. $\square$ claim

$\square$

Prop 3.3.12 Let R be a semiprime right Goldie ring. Then every essential right ideal is generated (as a right module) by regular elements.

Pf: The proof is based on a study of the connections between right ideals of R and of its quotient ring. For details see [McConnell-Robson, Section 3.3.7]. $\square$

Thm 3.3.13 (1) Suppose that R has a semisimple classical right quotient ring Q. Then Q is simple $\Leftrightarrow$ R is prime.
(2) TFAE: (a) R has a simple artinian classical right quotient ring.
(b) R is a prime right Goldie ring.
(3) If R is prime and right noetherian, then R has a simple artinian classical right quotient ring.

Pf: (1) Note that $R_R \subset Q_R$ is essential, since the product of any nonzero fraction with its denominator is a nonzero element of R.
$\Leftarrow$: Assume R prime, $I \neq 0$ an ideal in Q. Since $R_R \subset Q$ is essential, $I \cap R \neq 0$ in R. ideal of R!
R prime $\Rightarrow I \neq 0$ is essential
Then Lemma 2.3.3. implies that $I \cap R \subseteq_e R_R$ is essential, $\Rightarrow I \cap R \subseteq_e Q$.
Thus, $I_R \subseteq_e Q_R$ is essential and hence $I_Q \subset_e Q_Q$. Since Q is semisimple, this implies that $I = Q$. ($\leftarrow$ Lemma 2.3.4.(7)) $\Rightarrow Q$ is simple.
$\Rightarrow$: Suppose that Q is simple. Consider ideals $A, B \subset R$ with $AB = 0$ and suppose that $A \neq 0$. We have to show that $B = 0$, which implies that $0 \in \mathrm{Spec}(R)$.
Since $QAQ \subset Q$ is $\neq 0$, it follows that $QAQ = Q$. Q simple
Thus we can write

$$1 = p_1 a_1 q_1 + \dots + p_n a_n q_n, \text{ where } a_i \in A,\ p_i, q_i \in Q \text{ for } i \in \{1, \dots, n\}.$$

Let $b_1, \dots, b_n, x \in R$, with x regular s.t. $q_i = b_i x^{-1}$ $\forall i \in \{1, \dots, n\}$.

$$\Rightarrow \quad x = \Big(\sum_{i=1}^{m} p_i a_i q_i\Big) x = \sum_{i=1}^{n} p_i \underbrace{a_i b_i}_{\in A, \text{ since } A \text{ 2-sided ideal in } R} \in QA$$

$\Rightarrow xB \subset QAB = 0$. Since x is regular, it follows that $B = 0$.
(2) This follows from (1) and Thm 3.3.11.
(3) This follows from (2) and Thm 3.3.11. $\square$

Rmk 3.3.14 (1) By Thm 3.1.5, classical right quotient rings are unique up to isomorphism.

(2) If R is a right + left Goldie ring, then R is called <u>Goldie ring</u>.

Let R be a semiprime Goldie ring. By Prop 3.1.7, every classical left quotient ring is a classical right quotient ring and conversely. By (1) we speak of the <u>classical quotient ring</u>.

4. Further topics?

- Orders in quotient rings (Q is called quot. ring if every regular element is invertible, $R \subset Q$ is a <u>right order</u> in Q if each $q \in Q$ has the form rs^{-1} for some $r, s \in R$. Def. <u>left order</u> analogously ⇝ order = left + right order). ⇝ classically: study orders over integral domains in central simple algebras.
  Study rings of the form: $\mathrm{End}_R M$
- Morita theory
- Dimensions (Krull dim, global dimension, GK-dimension, ...)