Recall right quotient ring: $X \subset R$ mult closed, all elts of X regular

Then overring $Q \supset R$ is right quotient ring if

- $\forall x \in X: \ x^{-1} \in Q$
- Every elt of Q is of the form ax^{-1} for some $a \in R$, $x \in X$.

Thm of Ore-Asano shows: $\exists Q$ if X is right Ore set

$\forall x \in X \ \forall b \in R \ \ bX \cap xR \neq \emptyset$
$(\iff \exists z \in X, c \in R$ s.t. $bz = xc)$

+ universal property of Q.

We still have to prove the universal property of Q (2)+(3) of Thm 3.1.5.)

Recall: Thm 3.1.5 (Ore, Asano): Let $X \subseteq R$ be mult. closed subset of regular elements.

(1) $\exists$ right quotient ring of R w.r.t. $X \iff X$ is a right Ore set.

(2) Suppose that X is a right Ore set and $Q \supset R$ is a right quotient ring. Let $\varphi : R \to T$ be a ring homom., s.t. $\varphi(x)$ is invertible $\forall x \in X$. Then φ extends uniquely to a ring hom. $\psi : Q \to T$

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & T \\ Q \arrow[ur, "\psi"'] & \end{tikzcd}$$

(3) Let X be a right Ore set and Q, Q' be right quotient rings for R w.r.t. X. Then the identity $id_R : R \to R$ extends uniquely to an isom. of Q onto Q'.

Proof of (2)+(3):

(2) There is at most one possibility for such a map ψ, because

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, hook, "i"'] & T \\ Q \arrow[ur, "\psi"'] & \end{tikzcd}$$

$$\psi(ax^{-1}) \underset{\psi \text{ ring hom}}{=} \psi(a)\psi(x)^{-1} \underset{\text{ass: } \psi \text{ ext. } \varphi}{=} \varphi(a)\varphi(x)^{-1}.$$

Thus it suffices to show that such a ψ exists:

Let $a, b \in R$, $x, y \in X$ with $ax^{-1} = by^{-1}$. By Lemma 3.1.4 (4):

$\exists c,d \in R$ s.d. $ac=bd$ and $xc=yd \in X$. In part. $\varphi(x),\varphi(y)$ ②
$\varphi(xc),\varphi(yd)$ are all invertible in T, which implies $\varphi(c),\varphi(d) \in T^*$.
$\Rightarrow \varphi(a)\varphi(x)^{-1} = \varphi(ac)\varphi(xc)^{-1} = \varphi(bd)\varphi(yd)^{-1} = \varphi(b)\varphi(y)^{-1}$.
This means that φ is well defined for cosets $[a,x] \sim [b,y]$, and thus the rule $\psi(ax^{-1}) = \varphi(a)\varphi(x)^{-1}$ gives a well-def. map $\psi: Q \to T$.
Further $\psi(a) = \psi(a \cdot 1^{-1}) = \varphi(a)\varphi(1)^{-1} = \varphi(a)$ $\forall a \in R$, thus ψ extends φ, and $\psi(1) = 1$.
We have to check that ψ is a homomorphism: let $a,b \in R$, $x,y \in X$, then $\exists c,d \in R$, s.t. $xc=yd \in X$, $\Rightarrow$

$$\begin{aligned}\psi((ax^{-1})+(by^{-1})) &= \psi((ac+bd)(xc)^{-1}) = \varphi(ac+bd)\varphi(xc)^{-1}\\ &= \varphi(ac)\varphi(xc)^{-1} + \varphi(bd)\varphi(xc)^{-1}\\ &= \psi(ax^{-1}) + \psi(by^{-1}).\end{aligned}$$

Moreover, there are $e \in R$ and $z \in X$ s.t. $bz = xe$, thus

$$\begin{aligned}\psi((ax^{-1})(by^{-1})) &= \psi((ae)(yz)^{-1}) = \varphi(ae)\varphi(yz)^{-1}\\ &= \varphi(a)\varphi(e)\varphi(z)^{-1}\varphi(y)^{-1}.\end{aligned}$$

$ax^{-1}by^{-1}$
$= ax^{-1}bzz^{-1}y^{-1}$
$ae\,(yz)^{-1}$ ✓

Since $\varphi(b)\varphi(z) = \varphi(x)\varphi(e) \Rightarrow \varphi(e)\varphi(z)^{-1} = \varphi(x)^{-1}\varphi(b)$
and hence $\psi((ax^{-1})(by^{-1})) = \varphi(a)\varphi(x)^{-1}\varphi(b)\varphi(y)^{-1}$. $\Rightarrow \psi$ ring hom.
(3) Follows from (2). ψ unique! □

Since any two right quotient rings are isomorphic, we can make the following

<u>Def 3.1.6</u> If $X \subset R$ is a right Ore set of regular elements, then we write RX^{-1} for any right quotient ring of R w.r.t. X. Similarly, write $Y^{-1}R$ for a left quotient ring.

Prop 3.1.7 Let $X \subset R$ be a left + right Ore set of reg. elements. Then any right quotient ring (w.r.t. X) is also a left quot. ring (w.r.t. X), and conversely.

Pf: Let Q be a right quotient ring. Then Q is an overring of R and all elts of X are invertible in Q.
Let $s \in Q$, then $s = ax^{-1}$ for some $a \in R$, $x \in X$. Since X is a left Ore set, $\exists b \in R$ and $y \in X$ s.t. $ya = bx \Rightarrow s = y^{-1}b$. Thus Q is a left quotient ring. $\square$

https://www.jstor.org/stable/pdf/3647879.pdf#page23

§3.2. Classical quotient rings

Def 3.2.1 (1) A classical right quotient ring R is a right quotient ring for R w.r.t. $X = \{$regular elements$\}$.
If R has a classical right quotient ring Q then R is called a right order in Q. Classical left quotient rings Q are def. symmetrically, and in that case R is called a left order in Q.
(2) If a ring Q is a classical right quotient ring and a classical left quotient ring for R, then Q is called a classical quotient ring for R.
(3) The ring R is called a right Ore domain if the nonzero elements of R form a right Ore set ($\Leftrightarrow$: $\forall x,y \neq 0 \in R$, $\exists r,s \in R$ s.t. $xr = ys \neq 0$).

E.g.: (1) Every comm. ring has a classical quotient ring.
(2) Every comm. domain is an Ore domain and its classical quotient ring is its quotient field.

Lemma 3.2.2 For a domain ($\forall a,b \neq 0 \in R: a\cdot b \neq 0$) R TFAE:
(a) R is a right Ore domain.
(b) R_R is uniform
(c) R_R has uniform dimension $< \infty$.
In part: if R is right noetherian, then R is a right Ore domain.

Pf: (a) $\Leftrightarrow$ (b): The right Ore condition is equivalent to saying that the intersection of two nonzero principal right ideals is nonzero, and this is equivalent to saying that the right R-module R_R is uniform.
(b) $\Rightarrow$ (c): clear.
(c) $\Rightarrow$ (b): Assume on the contrary that the right R-module

R_R has finite uniform dim. but is not uniform.
Then $\exists$ nonzero ideals I_1 and J_1 of R s.t. $I_1 \cap J_1 = 0$. We choose an $x_1 \neq 0$ in J_1. Since R is a domain, we get $x_1 R \cong R$ (as right modules). Thus $x_1 R$ contains nonzero right ideals I_2 and J_2 s.t. $I_2 \cap J_2 = 0$,
and in part., we have $I_2 \oplus I_1 \subset R_R$. Proceeding by induction we obtain nonzero right ideals $I_1, J_1, I_2, J_2, \ldots$ s.t. $I_n \cap J_n = 0$ and $I_{n+1} + J_n \subset J_n$ $\forall n \geq 1$.
In part, we have $I_1 \oplus I_2 \oplus \ldots \oplus I_{n+1} \subset R$ $\forall n \geq 1$ ↯ to $\operatorname{udim}(R) < \infty$.
By Rmk 2.3.7., every right noetherian ring has finite uniform dim. $\square$

Thm 3.2.3 For a ring R, TFAE:

(a) There is a right Ore set X of regular elements in R s.t. RX^{-1} is a division ring.

(b) R has a classical right quotient ring which is a division ring.

(c) R is a right Ore domain.

Pf: The implication (b) $\Rightarrow$ (a) is clear.

(a) $\Rightarrow$ (b) The ring RX^{-1} is an overring of R whose elements have the form ax^{-1} with $a \in R$, $x \in X$. All regular elts of R are $\neq 0$ and since RX^{-1} is a division ring, they are invertible in RX^{-1}. Thus RX^{-1} is a classical right quotient ring of R (Defs 3.1.1 + 3.2.1)

(b) $\Rightarrow$ (c): Condition (b) implies that R is a domain and that all nonzero elements of R are regular. By Thm 3.1.5 (1), $R \setminus \{0\}$ is a right Ore set, which implies that R is a right Ore domain (by def.).

(c) $\Rightarrow$ (b) By Thm 3.1.5, R has a classical right quotient ring Q. Any element $\neq 0$ of Q has the form ax^{-1} for some nonzero elements $a, x \in R$. Since a is a regular element of R, it is invertible in $Q \Rightarrow ax^{-1}$ is invertible in Q. Thus Q is a division ring. $\square$ ⑥

3.3. Goldie's Theorem

Goldie's Thm provides necessary and sufficient conditions for a ring to have a semisimple right quotient ring. Alfred Goldie (1920-2005, Prof in Leeds, UK) introduced the concept of uniform dimension. See the article by Coutinho-McConnell for historical development.

Recall:

Def: Any left ideal I of R is called a left annihilator if it is the left annihilator of some subset S of R: $I := \{r \in R : r \cdot s = 0 \ \forall s \in S\}$ (similar: right annihilator)

Def 3.3.1 A ring R is called a right Goldie ring if R_R has finite uniform dimension and satisfies the ACC on right annihilators. ($\leadsto$ if R is noeth., the second cond. is automatic!)

Rmk 3.3.2 Let $I \subset R$ be a right ideal. Then I is a right annihilator if and only if $I = r\text{-ann}(\ell\text{-ann}(I))$.

Pf: By def $r\text{-ann}(\ell\text{-ann}(I))$ is a right annihilator. Conversely, suppose that I is a right annihilator, e.g., $I = r\text{-ann}(X) = \{r \in R : Xr = 0\}$ for some X in R. Then $X \subset \ell\text{-ann}(I)$, which implies

$$I = r\text{-ann}(X) \supset r\text{-ann}(\ell\text{-ann}(I)) \supset I. \quad \square$$

Prop 3.3.3 Suppose that R has a right noetherian classical quotient ring Q. Then R is a right Goldie ring. Moreover, if Q is semi-simple, then R is semiprime.

Pf: We proceed in 3 steps.

1. Show that R satisfies the ACC on right annihilators:
Let $I_1 \subset I_2 \subset \dots$ be an ascending chain of right annihilators in R. For $n \geq 1$ set $J_n := l\text{-ann}_R(I_n)$. This yields a descending chain of left annihilators.

If $r \in I_1 \Rightarrow r.I_1 = 0$
since $I_2 \subset I_1 \Rightarrow r.I_2 = 0$
$\Rightarrow r \in J_2$.

By rmk 3.3.2., it follows that $I_n = r\text{-ann}(J_n)\ \forall n \geq 1$.
In the ring Q, we have
$$r\text{-ann}_Q(J_1) \subset r\text{-ann}_Q(J_2) \subset \dots$$
is an AC of right ideals. Since Q is right noetherian, there is an $m \in \mathbb{N}_{\geq 1}$ s.t. $r\text{-ann}_Q(J_n) = r\text{-ann}_Q(J_m)$ $\forall n \geq m$. This implies that
$$I_n = R \cap r\text{-ann}_Q(J_n) = R \cap r\text{-ann}_Q(J_m) = I_m \quad \forall n \geq m.$$
$\Rightarrow$ R satisfies the ACC on right annihilators.

2. Assume on the contrary that R_R does not have finite uniform dimension. By Def 2.3.6, R contains an infinite direct sum of

$R \not\supseteq \bigoplus_{j=1}^{\infty} I_j$ submods = ideals

nonzero right ideals, say $R \supset \bigoplus_{n\geq 1} A_n$. For every $n \geq 1$ we choose a nonzero element $a_n \in A_n$. Since Q is right noetherian, the sum $\sum_{n\geq 1} a_n Q$ cannot be direct. (↯ to ACC)
$\Rightarrow$ (after possible renumbering) $\exists n \in \mathbb{N}$ and $q_1, \dots, q_n \in Q$ s.t.
$$a_1 q_1 + \dots + a_n q_n = 0 \text{ and all summands are } \neq 0.$$

Further, there exist $b_1, \dots, b_n, x \in R$, with x regular, s.t. $q_i = b_i x^{-1}$ $\forall i \in \{1, \dots, n\}$. Then
$$a_1 b_1 + \dots + a_n b_n = (a_1 q_1 + \dots + a_n q_n) x = 0.$$

Since the sum of $A_1, \dots, A_n$ is direct, it follows that $a_1 b_1 = \dots = a_n b_n = 0$

$\Rightarrow a_i\varphi_i = a_i b_i x^{-1} = 0 \quad \forall i \in \{1,\dots,n\}$, ↯. (all summands $\neq 0$!)

3) Suppose that Q is semisimple. We have to show that the 0 ideal in R is semiprime. ← i.e. $0 = \sqrt{0}$.

by Lemma 2.2.3.(d): $0 = \sqrt{0}$ $\Leftrightarrow$ if $N^2 = 0 \Rightarrow N = 0$.

Let $N \subset R$ be an ideal s.t. $N^2 = 0$. We need to show that $N = 0$.
The left annihilator $L := \ell\text{-ann}_R(N)$ is an ideal of R. By Lemma 2.3.3.(2): $L \subseteq_e R$ is essential.
We claim that $LQ \subset Q$ is an essential right ideal: Let $B \subset Q$ be a nonzero right ideal, choose a nonzero element $b \in B$ and set $b = ax^{-1}$ with $a, x \in R$ and x regular. Then

$$a = bx \in R \cap B \Rightarrow R \cap B \subseteq R \text{ is a nonzero right ideal.}$$

$\Rightarrow (R \cap B) \cap L \neq 0$ and hence $B \cap LQ \neq 0$. Thus $LQ \subseteq_e Q$ is ess.
Since Q_Q is semisimple, Lemma 2.3.4.(7) implies that $LQ = Q$. (only ess. mod of s.s. M is M)
$\Rightarrow \exists y_1, \dots, y_n \in L$ and $\varphi_1, \dots, \varphi_n \in Q$ s.t.

$$1 = \sum_{i=1}^n y_i \varphi_i .$$

Moreover, $\exists a_1, \dots, a_n, x \in R$ with x regular, s.t. $\varphi_i = a_i x^{-1}$ $\forall i \in \{1,\dots,n\}$.
Thus we obtain

$$x = (y_1\varphi_1 + \dots + y_n\varphi_n)x = y_1 a_1 + \dots + y_n a_n \in L.$$

Since $L = \ell\text{-ann}_R(N)$, it follows that $xN = 0$.
Since x is regular $\Rightarrow$ $N = 0$. $\square$

In part, this proposition means that any ring having a classical right quotient ring must be a semiprime right Goldie ring.
Our next goal is to prove the converse, which is the main content of Goldie's thm (Thm 3.3.11).

__Def__ 3.3.4: For a right module M, define

$$\begin{aligned} \operatorname{Sing}(M) &:= \{x \in M : xI = 0 \text{ for some essential } I \subseteq_e R_R\} \\ &= \{x \in M : r\text{-ann}_R(x) \subseteq_e R_R\}. \end{aligned}$$

$\mathrm{Sing}(M)$ is called the <u>singular submodule</u> of M.
If $M = R_R$, then $\mathrm{Sing}(R)$ is called the <u>singular ideal</u> of R.
If $\mathrm{Sing}(R) = 0$, then R is called <u>non-singular</u>.

<u>Ex</u>: Set $R = k[x,y]/(xy)$, then $\mathrm{Sing}(R) = (0)$ (!)
$R = k[x]/(x^2)$, then (x) is essential and $\mathrm{Sing}(R) = (x)$.

<u>Caution</u>: This is not the standard meaning of "$\mathrm{Sing}(R)$" from alg. geometry / commutative algebra! [$\mathrm{Sing}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R)$ s.t. $R_{\mathfrak{p}}$ is not regular$\}$]

<u>Lemma 3.3.5.</u> (1) If M is a right R-module, then $\mathrm{Sing}(M) \subset M$ is a submodule.
(2) $\mathrm{Sing}(R) \subset R$ is a two-sided ideal.

<u>Pf</u> (1) Clearly, $\mathrm{Sing}(M)$ is an additive subgroup (if $xI = yJ = 0$ $\Rightarrow (x+y)(I \cap J) = 0$ since $I \cap J \subseteq_e R$)
We have to show that it is stable under right scalar mult.;
Set $x \in \mathrm{Sing}(M)$ and $s \in R$. By Prop 2.3.4. (3) ($N \subseteq_e M$, $m \in M$ $\Rightarrow m^{-1}N \subseteq_e R_R$)

$$s^{-1}(r\text{-}\mathrm{ann}_R(x)) = \{r \in R : sr \in r\text{-}\mathrm{ann}_R(x)\} \subseteq_e R_R$$

is essential in and clearly

$$xs \cdot s^{-1}(r\text{-}\mathrm{ann}_R(x)) \subset x\,(r\text{-}\mathrm{ann}(x)) = 0.$$

$\Rightarrow s^{-1}(r\text{-}\mathrm{ann}(x)) \subset r\text{-}\mathrm{ann}(xs)$. Since $s^{-1}(r\text{-}\mathrm{ann}_R(x)) \subseteq_e R_R$ is ess.
$\Rightarrow r\text{-}\mathrm{ann}(xs) \subseteq_e R_R$ is essential $\Rightarrow xs \in \mathrm{Sing}(M)$.
(2) By (1) we get that $\mathrm{Sing}(R) \subset R$ is a right ideal. Set $x \in \mathrm{Sing}(R)$ and $r \in R$. Since $r\text{-}\mathrm{ann}_R(rx) \supset r\text{-}\mathrm{ann}_R(x) \Rightarrow r\text{-}\mathrm{ann}_R(rx) \subseteq_e R$ is essential $\Rightarrow rx \in \mathrm{Sing}(R)$. $\square$

<u>Prop 3.3.6.</u> (1) If R satisfies the ACC on right annihilators, then $\mathrm{Sing}(R)$ is nilpotent.
(2) If R is a semiprime right Goldie ring, then the left annihilator of an essential right ideal is 0.

Pf: (1) Set $J := \mathrm{Sing}(R)$. We have

$$r\text{-ann}(J) \subset r\text{-ann}(J^2) \subset \dots$$

$J^2 \subset J$, so if $xr = 0$ for all $x \in J$, then $y(xr) = 0 \Rightarrow J^2 r = 0$ $\forall y \in J$
(i.e. $r \in r\text{-ann}(J)$),

Since R satisfies the ACC on right annihilators, $\exists k \in \mathbb{N}$ s.t. $r\text{-ann}(J^k) = r\text{-ann}(J^{k+1})$ and we claim that $J^k = 0$.
Assume on the contrary that $J^k \neq 0$, and choose an $x \in R \setminus r\text{-ann}(J^{k+1})$, such that $r\text{-ann}(x)$ is maximal. Now, for every $a \in J$, $r\text{-ann}(x) \subseteq_e R_R$ is essential, which implies $r\text{-ann}(x) \cap xR \neq 0$.
$\Rightarrow \exists s \in R$ s.t. $axs = 0$ and $xs \neq 0$. Then $r\text{-ann}(x) \subsetneq r\text{-ann}(xs)$, and the maximality of $r\text{-ann}(x)$ implies that $ax \in r\text{-ann}(J^k)$.
Thus $J^k ax = 0$. Since this holds $\forall a \in J$, we obtain that $x \in r\text{-ann}(J^{k+1}) = r\text{-ann}(J^k)$. ↯

(2) Note that any left annihilator $l\text{-ann}(I)$ of an essential right ideal $I \leq R$ lies in $\mathrm{Sing}(R)$. Since a right Goldie ring satisfies ACC on right annihilators, $\mathrm{Sing}(R)$ is nilpotent by (1). Since R is semiprime, Cor 2.2.7 $\Rightarrow$ $l\text{-ann}(I) \subset \mathrm{Sing}(R) = 0$.
R semip. $\Rightarrow$ R has no $\neq 0$ nilp. id.

$\square$