Recall: We are in the process of proving Cor 2.3.10 ((4) still missing!)

Cor 2.3.10 Let M be an R-module.

- M uniform $:\Leftrightarrow$ every $\neq 0$ submodule of M is essential
- $\mathrm{udim}(M) \overset{2.3.9}{=} \max_n \{ \bigoplus_{i=1}^{n} U_i \leq_e M,\ U_i \neq 0$ all uniform $\}$

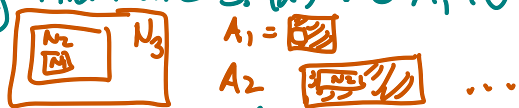
(1) $\mathrm{udim}(M) = 1 \Leftrightarrow M$ is uniform.
$\mathrm{udim}(M) = 0 \Leftrightarrow M = 0$.
(2) Let $\mathrm{udim}(M) = n$ and $N \subset M$. Then $\mathrm{udim}(N) = n \Leftrightarrow N \subseteq_e M$ is essential.
(3) $\mathrm{udim}(M_1 \oplus M_2) = \mathrm{udim}(M_1) + \mathrm{udim}(M_2)$.

Proven

(4) M has finite uniform dimension $\Leftrightarrow M$ satisfies the ACC for complement submodules. Indeed, $\mathrm{udim}(M) =$ max. length of a chain of complement submodules.

$N \subset M$ s.t. $\exists N' : N \oplus N' = M$

Proof of (4):
If $N_1 \subsetneq \dots \subsetneq N_t$ is a properly ascending chain of complement submodules, then by Rmk 2.3.5 (2): $\exists A_i \neq 0$ in N_i s.t. $A_i \cap N_{i-1} = 0$ for all $i \in \{2, \dots, t\}$.



Then $N_1, N_1 \oplus A_2, \dots, N_1 \oplus A_2 \oplus \dots \oplus A_t$ are submodules of M.
Thus, if $\mathrm{udim}(M) = n$, then there is no chain of complement modules of length $> n$.
Conversely, if $M \supseteq \bigoplus_{i=1}^{t} M_i$, where $t \in \mathbb{N} \cup \{\infty\}$, then the complements of the submodules $\bigoplus_{i=n}^{t} M_i$, where $n \in \{1, \dots, t\}$, form a properly ascending chain of length t. $\square$

For the following two results, we need (R,R)-bimodules: Consider a module ${}_RM_R$. Then ${}_RM_R$ can be considered as a right $R \otimes R^{op}$-module, and we can use all our former results on right modules. Then, to say that the bimodule ${}_RM_R$ has finite uniform dimension means that $M_{R \otimes R^{op}}$ has finite uniform dimension, and that means that M contains no infinite direct sum of nonzero sub-bimodules. We use this point of view for two-sided ideals and for the ring $R = {}_RR_R$.

Thm 2.3.11 Let R be semiprime and $I \subseteq R$ be an ideal.

(1) $\underbrace{l\text{-}ann_R(I)}_{\{\lambda \in R \,:\, \lambda \cdot x = 0 \ \forall x \in I\}} = \underbrace{r\text{-}ann_R(I)}_{\{\lambda \in R \,:\, x \cdot \lambda = 0 \ \forall x \in I\}}$, and we set $ann_R(I) := l\text{-}ann_R(I)$.

(2) $ann_R(I)$ is the unique complement of I in R.

(3) $ann_R(I) = \bigcap$ of all minimal prime ideals in R which do not contain I.

(4) I is uniform $\iff$ $ann_R(I)$ is a minimal prime ideal.

(5) $I \leq_e R$ is essential (as a left and as a right ideal/module) $\iff$ $ann_R(I) = 0$.

(6) If I is not contained in any minimal prime ideal, then $I \leq_e R$ is essential.

Cor 2.2.7: R semiprime $\Rightarrow$ R has no nilpotent nonzero ideal.

Pf: (1) If $X \subset R$ and $I \cdot X = 0$, then $(XI)^2 \overset{X\overset{0}{\overbrace{IX}}I}{=} 0$, which implies $XI = 0$. Conversely, if $XI = 0 \Rightarrow IX = 0$. Thus $l\text{-}ann_R(I) = r\text{-}ann_R(I)$.

(2) Show that $ann_R(I)$ is the unique left complement module of I in R, and $ann_R(I)$ is the — " — right — " — :

If $J \subseteq R$ is a left ideal with $I \cap J = 0$, then $IJ \subset I \cap J = 0$, which implies that $J \subset ann_R(I)$. The same argument works for right ideals. On the other hand $(I \cap ann_R(I))^2 = (I \cap ann_R(I)) \cdot (I \cap ann_R(I)) = 0$, and this implies (Cor 2.2.7) that $I \cap ann_R(I) = 0$. Thus $ann_R(I)$ is the uniquely determined complement module. (← since it is max J with the property $I \cap J = 0$)

(3) Let $J :=$ intersection of all min. prime ideals of R not containing I.

Since R is semi-prime $\Longrightarrow$ $0 \overset{\text{Cor 2.2.7}}{=} \sqrt{0} = \bigcap_{\substack{\wp \subseteq R \\ \text{prime}}} \wp = \Big(\underbrace{\bigcap_{I \subseteq \wp} \wp}\Big) \cap J = \underbrace{\sqrt{I}}_{\text{Lemma 2.2.2.(2)}} \cap J \supseteq I \cap J \supseteq I \cdot J$.

Thus $J \subseteq ann_R(I)$ by (2).

For the other inclusion, note that $ann_R(I) \cdot I = 0$ ($ann_R(I) \cap I = 0$). If $\wp$ is a minimal prime ideal and $I \not\subseteq \wp$, then $ann_R(I) \subset \wp$. Thus $ann_R(I) \subseteq J$.

(4) $\Longrightarrow$: Suppose that I is uniform. By (2), $ann_R(I)$ is the unique

complement of I in R.

$I' \subseteq I$ $\quad x \in \mathrm{ann}_R(I) \Rightarrow x \cdot a = 0 \;\forall a \in I \Rightarrow x \cdot a' = 0 \;\boxed{\forall a' \in I' \subseteq I}$
$\Rightarrow \mathrm{ann}_R(I) \subseteq \mathrm{ann}_R(I')$

<u>Claim</u>: $\mathrm{ann}_R(I') = \mathrm{ann}_R(I) \;\forall I' \neq 0 \subseteq I$.

$\therefore$ We have to show that $\mathrm{ann}_R(I') \cap I = 0$. ($\Rightarrow \mathrm{ann}_R(I') \subseteq \mathrm{ann}_R(I)$)

Assume on the contrary that $\mathrm{ann}_R(I') \cap I \neq 0$, then, since I' is uniform too, $I' \cap (\mathrm{ann}_R(I') \cap I) \neq 0$ but $I' \cap \mathrm{ann}_R(I') = 0$. ↯ $\square$ Claim

Let now $B, C \subseteq R$ be ideals with $BC \subseteq \mathrm{ann}_R(I)$. Then, either (i) $IB = 0$ or (ii) $IB \neq 0$.

In case (i): $B \subseteq \mathrm{ann}_R(I)$

In case (ii): $C \subset \mathrm{ann}_R(IB) = \mathrm{ann}_R(I)$.

Thus $\mathrm{ann}_R(I)$ is a prime ideal, and minimality follows from (3).

$\Leftarrow$: Suppose that $\mathrm{ann}_R(I)$ is a min. prime ideal. If $J \subset I$ is a nonzero ideal, then $\mathrm{ann}_R(J) \supseteq \mathrm{ann}_R(I)$. Since $\mathrm{ann}_R(J)$ is the intersection of those minimal prime ideals not containing J, it follows that $\underline{\mathrm{ann}_R(I) = \mathrm{ann}_R(J)}$.

If $J_1, J_2 \subseteq R$ are nonzero ideals with $J_1 \oplus J_2 \subset I$, then $\mathrm{ann}_R(I) \subsetneq \mathrm{ann}_R(J_1)$. ↯ to def Thus I does not contain a direct sum of nonzero ideals $\Rightarrow$ I is uniform.

(5) Follows from (2).

(6) $\sqrt{0} = \bigcap_{\wp \in \mathrm{Spec}(R)} \wp$, and since R is semiprime, we have $\underline{\sqrt{0} = 0}$ (by Cor 2.2.7). Thus, if I is not contained in any min. prime, (3) says that $\mathrm{ann}_R(I) = \bigcap_{\text{min. primes}} \wp = 0$. Then (5) shows the claim. $\square$

<u>Cor 2.3.12</u> Let R be a semiprime ring. TFAE:

(a) R has finite uniform dimension.

(b) R has finitely many min. prime ideals.

(c) R has finitely many annihilator ideals.

(d) R satisfies the ACC on annihilator ideals.

Pf: (a) ⇒ (b) Let $U_1, \dots, U_n$ be uniform ideals s.t. $U_1 \oplus \dots \oplus U_n \subseteq_e R$. For $i \in \{1, \dots, n\}$ we set $\mathfrak{p}_i = \mathrm{ann}_R(U_i)$. By Thm 2.3.11. (4), $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are minimal prime ideals. By 2.3.11. (5) $\overset{I \subseteq_e R \Leftrightarrow \mathrm{ann}_R(I) = 0}{\Rightarrow}$

$$0 = \mathrm{ann}_R(U_1 \oplus \dots \oplus U_n) = \bigcap_{i=1}^n \mathfrak{p}_i .$$

If $\mathfrak{p}$ is a min. prime, then $\mathfrak{p}_1 \cdots \mathfrak{p}_n \overset{\subset \bigcap \mathfrak{p}_i = 0}{\subset} \mathfrak{p} \Rightarrow \mathfrak{p} \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.

(b) ⇒ (c) Thm 2.3.11. (3)

(c) ⇒ (d): Clear (every finite chain becomes stationary!)

(d) ⇒ (a): By Thm 2.3.11. (2): $\mathrm{ann}_R(I)$ = unique complement of I in R. Thus by Cor 2.3.10 (4), R has finite uniform dimension. □

[MC-R, p56]

Cor 2.3.13 If R is a semiprime ring with $\mathrm{udim}_R R_R = n$, then R has n min. primes, 2^n annihilator ideals, any max. chain of annihilator ideals has length n and no min. prime is essential in $_R R_R$

Pf (sketch): The $\mathfrak{p}_i$ from the proof above are all the min. primes.
Further, since $\mathfrak{p}_{i_1} \cap \dots \cap \mathfrak{p}_{i_k} = \mathrm{ann}_R(U_{i_1} \oplus \dots \oplus U_{i_k})$
⇒ each intersection of min. primes is an annihilator.
$\overset{2.3.11\,(3)+(4)}{\Longrightarrow}$ complete proof. □

Example (commutative) Let $R = k[x,y]/(xy)$. Then R is semiprime since $\sqrt{\bar{0}} = \bar{0} = (\bar{x}) \cap (\bar{y})$, and those are the min. primes [$(\bar{x})$ prime since $R/(\bar{x}) \cong k[y]$ is an int. domain, same argument for $(\bar{y})$]

$\mathrm{ann}_R(\bar{x}) = (\bar{y})$ and $\mathrm{ann}_R(\bar{y}) = (\bar{x})$ and $U_1 = (\bar{x})$, $U_2 = (\bar{y})$.
$U_1 \oplus U_2 \subset R$ and $\mathrm{udim}(R) = 2$

The possible complement submodules of R: $(\bar{x}), (\bar{y})$, $\underset{\mathrm{ann}(\bar{0})}{R}$, $\underset{\text{compl. of } \bigoplus_{i=1}^n U_i}{(0)}$

Chains: $(0) \subseteq (\bar{x}) \subseteq R$ $\qquad$ $\mathrm{udim}(R) = 2$. □
$(0) \subseteq (\bar{y}) \subseteq R$.

CHAPTER 3: QUOTIENT RINGS AND GOLDIE'S THEOREM

Goal is now to localize noncomm. rings

goldie: R is Goldie ring $\Leftrightarrow$ R has s.s. quot. ring

Recall construction of localization in commutative case:

Let R be a comm. ring. $S \subseteq R$ is <u>multiplicatively</u> <u>closed</u> if:

- $1_R \in S$, • $\forall s, s' \in S: s \cdot s' \in S$.

<u>Localization</u>: $S^{-1}R$ (or $R[S^{-1}]$): $S^{-1}R = \{(a,s) \in R \times S\}/\sim$

where $(a,s) \sim (a',s') :\Leftrightarrow \exists t \in S: t(as' - a's) = 0$

$S^{-1}R$ is a comm. ring with $(a,s) + (a',s') := (as' + a's, ss')$ $\quad \frac{a}{s} + \frac{a'}{s'} = \dots$

$(a,s) \cdot (a',s') := (aa', ss')$ $\quad \frac{a}{s} \cdot \frac{a'}{s'}$

<u>Ex</u>: (1) $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\} \Rightarrow S^{-1}R = \mathbb{Q}$ ← field: simple

(2) R comm., $\mathfrak{p} \subseteq R$ prime id, set $S := R \setminus \mathfrak{p}$, then $R_{\mathfrak{p}} := \{\frac{a}{s} : s \in R \setminus \mathfrak{p}\}/\sim$ (local ring with max. id. $\mathfrak{p}R_{\mathfrak{p}}$; $a \in R$)

(3) R comm. $f \neq 0 \in R$, take $S = \{f, f^2, f^3, \dots\}$ $\quad R_f = \{\frac{a}{f^i}, a \in R\}$

$R = k[x]$, $f = x \Rightarrow R_x = k[x, x^{-1}]$

$R = k[x]$, $\mathfrak{p} = (x) \Rightarrow R_{(x)} = \{\frac{P}{Q}, Q(0) \neq 0\}$.

$S \subseteq R$ mult. closed, $\varphi: R \to S^{-1}R$, $r \mapsto r/1$
$\Rightarrow \varphi(R) \subseteq S^{-1}R$ is invert.
<u>UP</u>: $R \xrightarrow{f} B$ any ringhom s.t. $f(s) \in B^* \ \forall s \in S$
$\Rightarrow \exists! h: S^{-1}R \to B$ ringhom s.t. $f = h \circ \varphi$

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\varphi"'] & B \\ S^{-1}R \arrow[ur, dashed, "\exists! h"'] & \end{tikzcd}$$

Can describe localization via <u>universal property</u>.

In particular: $Q(R) = \{\frac{a}{s} : s \text{ NZD} \in R\}$ total ring of quotients

If R is an integral domain, then $Q(R)$ is a field, the smallest field in which R can be embedded (any element of $Q(R)$ is of the form ab^{-1}, $b \neq 0$, or any elt $\neq 0$ is invertible)

$\rightsquigarrow$ Can we do this constructions for noncommutative rings?

Let now R be any ring (always with 1), then define $X \subseteq R$ to be <u>mult. closed</u> if X is a <u>subsemigroup of</u> $(R, \cdot)$

(same def as in comm case: $1_R \in X$ and X closed under $\cdot$)

<u>Rmk</u>: $R^\circ = \{\text{reg. elts of } R\} = \{x \in R : x \text{ not a left or right zero div}\}$

§ 3.1. Quotient rings

Let $R \subset Q$ be a ring extension and let $a \in Q^*$. Then a is regular in R ($\because$ if $ax = 0$ for some $x \in R \Rightarrow x = 1_Q \cdot x = (a^{-1} \cdot a)x = a^{-1}(\overset{=0}{ax}) = 0$, similarly for $xa = 0$)

Thus we can assume w.l.o.g. that the set X in the next def. consists of regular elements:

<u>Def 3.1.1.</u> Let $X \subseteq R$ be a mult. closed subset of regular elements. An overring $Q \supset R$ is called a <u>right quotient ring</u> for R with respect to X if the following hold:

(a) Every element of X is invertible in Q.

(b) Every element of Q can be expressed as ax^{-1} for some $a \in R, x \in X$.

<u>Ex:</u> - R int. domain, $X = R \setminus \{0\} \rightsquigarrow Q = X^{-1}R$
- $k[x, x^{-1}] = \{x, x^2, \dots\}^{-1} k[x]$ is right quot. ring, but not a field

<u>Rmk 3.1.2</u> Now we expect computations in rings of fractions to behave like fractions in the commutative case:

1. addition of fractions: define $ax^{-1} + by^{-1}$?
 Clearly, if $ax^{-1} = a'z^{-1}$ and $by^{-1} = b'z^{-1}$, then define
$$ax^{-1} + by^{-1} := (a' + b')z^{-1}.$$
 Note that $ax^{-1} = (ay)(xy)^{-1}$ but in general $by^{-1} \neq bx(xy)^{-1}$.
 But all we need is an element $z \in X$, s.t. $z = xc = yd$ for some $c, d \in R$, since then $ax^{-1} = (ac)z^{-1}$ and $by^{-1} = (bd)z^{-1}$.

2. Products of fractions: here we get into trouble:
 in general we cannot expect that $(ax^{-1})(by^{-1}) = (ab)(yx)^{-1}$, unless a and b commute.
 However, since any elt. of Q is of the form cz^{-1} (by assumption) with $c \in R, z \in X$, $x^{-1}b = cz^{-1} \Rightarrow bz = xc \in bX \cap xR$.

Then: $(ax^{-1})(by^{-1}) = a(cz^{-1})y^{-1} = (ac)(yz)^{-1}$.

This will be used in the Ore condition: about ∃ of rings of fractions

Def 3.1.3 Let $X \subseteq R$ be a mult. closed set.

(1) X satisfies the right Ore condition (or X is a right Ore set) if $\forall x \in X$ and $\forall b \in R$: $bX \cap xR \neq \emptyset$ ($\Leftrightarrow \exists z \in X$ and $\exists c \in R$ s.t. $bz = xr$).

(2) X satisfies the left Ore condition if $\forall x \in X$ and $\forall b \in R$: $Xb \cap Rx \neq \emptyset$ ($\Leftrightarrow \exists z \in X, c \in R: zb = cx$).

(3) An Ore set is a mult. closed set which is both a left and right Ore set.

Example [McC-R p.44, 1.7] Let $R = k\langle y_1, y_2\rangle$ the free assoc. algebra in two vars over a field k. Let $X = R\setminus\{0\}$. The right Ore condition fails, e.g. choose $b = y_1$, $x = y_2$ ⇝ $y_1 \cdot P = y_2 Q$ ∄ for $P, Q \neq 0$ not comm.! no relat.

⇝ We will see that then quot. ring Q_X does not exist!

Lemma 3.1.4 Let $X \subseteq R$ be a mult. closed subset of regular elements and Q a right quotient ring (of R w.r.t. X).

(1) X is a right Ore set.

(2) Let $n \geq 1$ and $x_1, \dots, x_n \in X$. Then $\exists s_1, \dots, s_n$ with $x_1 s_1 = \dots = x_n s_n \in X$ (i.e. $x_1 R \cap \dots \cap x_n R \cap X \neq \emptyset$).

(3) Let $n \geq 1$ and $s_1, \dots, s_n \in Q$. Then $\exists a_1, \dots, a_n \in R$ and $x \in X$ s.t. $s_i = a_i x^{-1}$ $\forall i \in \{1, \dots, n\}$.

(4) Let $a, b \in R$ and $x, y \in X$. Then $ax^{-1} = by^{-1} \Leftrightarrow \exists c, d \in R$ s.t. $ac = bd$ and $xc = yd \in X$.

(5) For every R-module M,

$t_X(M)$ is called <u>torsion submodule w.r.t. X</u>

$$t_X(M) := t(M) = \{m \in M : mx = 0 \text{ for some } x \in X\} \subset M$$

is a submodule. In part., $t_X(R_R) \subset R$ is a two-sided ideal.

<u>Pf</u> (1) See Rmk 3.1.2.

(2) By induction, it is sufficient to prove the case $n=2$.
By the right Ore condition, $\exists y \in X$ and $s \in R$ s.t. $x_1 y = x_2 s$.
Since X is mult. closed, it follows that $x_1 y \in X$.

(3) We set $s_i = b_i x_i^{-1}$ with $b_i \in R$, $x_i \in X$ $\forall i \in \{1, \dots, n\}$. By (2), $\exists x \in X$ and $c_1, \dots, c_n \in R$ with $x = x_i c_i$ $\forall i \in \{1, \dots, n\}$.
Since $x, x_i \in X \subset Q^*$, it follows that $c_i = x_i^{-1} x \in Q^* \Rightarrow$
$x^{-1} = c_i^{-1} x_i^{-1}$ and

$$s_i = b_i x_i^{-1} = \underbrace{b_i c_i}_{=:a_i} x^{-1} \quad \forall i \in \{1, \dots, n\}.$$

(4) $\Leftarrow$: If $\exists c, d \in R$ with $ac = bd$ and $xc = yd \in X$, then

$$a x^{-1} = ac(xc)^{-1} = bd(yd)^{-1} = b y^{-1}.$$

$\Rightarrow$: Conversely, suppose that $a x^{-1} = b y^{-1}$. By (2) $\exists c, d \in R$ with $xc = yd \in X$. Thus we obtain

$$ac(xc)^{-1} = a x^{-1} = b y^{-1} = bd(yd)^{-1} = bd(xc)^{-1}.$$

$\Rightarrow ac = bd$.

(5) Let $m_1, m_2 \in t(M)$. Then $\exists x_1, x_2 \in X$ s.t. $m_1 x_1 = m_2 x_2 = 0$.
By (2): $\exists y \in x_1 R \cap x_2 R \cap X$ with $(m_1 \pm m_2) y = 0$. $\Rightarrow m_1 \pm m_2 \in t(M)$.
If $r \in R$, then by the right Ore condition, $\exists z \in X$ and $s \in R$ with $rz = x_1 s$. Thus $m_1 r z = m_1 x_1 s = 0 \Rightarrow m_1 r \in t(M)$.
$\Rightarrow t(M)$ is a right R-module. $\square$

<u>Thm 3.1.5</u> (Ore, Asano): Let $X \subseteq R$ be mult. closed subset

of regular elements.

(1) $\exists$ right quotient ring of R w.r.t. $X \iff X$ is a right Ore set.

(2) Suppose that X is a right Ore set and $Q \supset R$ is a right quotient ring. Let $\varphi: R \to T$ be a ring homom., s.t. $\varphi(x)$ is invertible $\forall x \in X$. Then φ extends uniquely to a ring hom. $\psi: Q \to T$

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & T \\ Q \arrow[ur, "\psi"'] & \end{tikzcd}$$

(3) Let X be a right Ore set and Q, Q' be right quotient rings for R w.r.t. X. Then the identity $id_R: R \to R$ extends uniquely to an isom. of Q onto Q'.

Pf: (1) $\Rightarrow$: This is Lemma 3.1.4.(1).

$\Leftarrow$: Let X be a right Ore set. We sketch an elementary construction by Ore and Asano. (Note: another, slicker way, would be to consider the injective hull $E(R_R)$ of R, define a suitable submodule $A \subset E(R_R)$, and show that $End_R(A)$ satisfies the properties of the quotient ring. See e.g. [Goodearl-Warfield, Thm 6.2 p.108])

tedious (as in comm. case $\rightsquigarrow$ only sketch)

It proceeds in 5 steps:

(i) Define a relation $\sim$ on $R \times X$ as follows:

$$(a,x) \sim (b,y) :\iff \exists c,d \in R \text{ s.t. } ac = bd \text{ and } xc = yd \in X.$$

This is an equivalence relation (!) and for any pair $(a,x) \in R \times X$, let $[a,x]$ denote its equiv. class. Define $Q := \{[a,x] : a \in R, x \in X\}$.

(ii) Let $[a,x], [b,y] \in Q$. Choose $c,d \in R$ s.t. $xc = yd \in X$ and set $[a,x] + [b,y] := [ac+bd, xc]$. This is a well-def. operation on Q (!)

(iii) Let $[a,x], [b,y] \in Q$. Choose $c \in R$, $z \in X$ s.t. $bz = xc$, and set $[a,x] \cdot [b,y] = [ac, yz]$ $\quad ax^{-1}by^{-1} = ax^{-1}(xcz^{-1})y^{-1} = ac(yz)^{-1}$

This is a well-defined operation on Q. (!)

(iv) $(Q, +, \cdot)$ is a ring (!).

(v) The map $i : R \to Q : r \mapsto [r, 1]$ is a ring monomorphism. When R is identified with $i(R)$, then Q becomes the quotient ring of R w.r.t. X.