WEEK 12

Recall: Prime ring, prime radical
Semi-prime ring → Lemma 2.2.3 / Cor 2.2.7

- $\wp \subseteq R$ two-sided ideal is prime $\Leftrightarrow$ (1) $\wp \subsetneq R$, (2) $\forall I, J \subseteq \wp$: $I \cdot J \subseteq \wp \Rightarrow I \subseteq \wp$ or $J \subseteq \wp$.

- R is prime $\Leftrightarrow$ (0) is a p. ideal.
- R is semi-prime $\Leftrightarrow$ $(0) = \bigcap_{\substack{\wp \subseteq R \\ \text{prime}}} \wp$ (Lemma 2.2.3: I is semi-prime if $I = \sqrt{I} = \bigcap_{\substack{\wp \supseteq I \\ \wp \text{ prime}}} \wp$) → def via m-system

- $\sqrt{0}$ is prime radical = nil radical of R
  Have: $\sqrt{0} \subseteq \{s \in R,\ s \text{ nilpotent, i.e. } \exists n \geq 0 : s^n = 0\}$.

- Have always: $\sqrt{0} \subseteq J(R) = \bigcap_{m \subseteq R \text{ max.}} m$
- Cor 2.2.7: R semi-prime $\Leftrightarrow$ $\sqrt{0} = 0$.

Thm 2.2.8

(1) If R is left artinian, then $\sqrt{0} = J(R)$.

(2) TFAE: (a) R is semi-simple
(b) R is left artinian and $J(R) = 0$.
(c) R is left artinian and has no nonzero nilpotent ideals.
(d) R is left artinian and semiprime.

(3) TFAE: (a) R is simple and left artinian.
(b) R is simple and semi-simple.
(c) R is isomorphic to a matrix ring over a division ring.
(d) R is prime an left artinian.

(4) If R is left artinian, then every prime ideal is maximal. (2)

Pf: By Thm 1.4.9. $J(R)$ is nilpotent $\Rightarrow J(R) \overset{1.4.3}{\subset} \sqrt{0} \subset J(R)$ (note that this statement would also follow from (4)).

(2) follows from Thm 1.4.10 (R s.s. $\Leftrightarrow R$ art. $+ J(R)=0$) and Cor 2.2.7. R simple $\Rightarrow R$ left art. R s.s. $\Downarrow$ $R \cong M_n(D)$

(3) The equivalence of (a), (b), and (c) follows from Cor 1.3.16.

(a) $\Rightarrow$ (d): Every simple ring is prime. ((0) is only p.i.)

(d) $\Rightarrow$ (b): Let R be prime and left artinian. Then R is semi-simple by 2, and by Thm 1.3.10 a finite direct sum of simple rings, say $R = R_1 \times \dots \times R_s$ with $s \geq 1$. Since the product of two or more non-zero rings cannot be prime $((0, I_2) \cdot (I_1, 0) = (0,0) \overset{\text{ideal } \neq 0}{\subseteq} R_1 \times R_2$ for any $I_i \trianglelefteq R_i)$, it follows that $s = 1$, i.e., R is simple.

(4) Let R be left artinian. Assume on the contrary that there is a non maximal prime ideal $\mathfrak{p} \subsetneq R$ ($\mathfrak{p} \subsetneq \mathfrak{m} \subsetneq R$). Then $R/\mathfrak{p}$ is a left artinian prime ring which is not simple (since $0 \subsetneq \mathfrak{m}/\mathfrak{p} \subsetneq R/\mathfrak{p}$ is ideal) $\lightning$ to (3). $\square$

Lemma 2.2.9 Let $n \in \mathbb{N}_{\geq 1}$.

(1) R is prime (semiprime) $\Leftrightarrow M_n(R)$ is prime (semi-prime).

(2) $\sqrt{0_{M_n(R)}} = M_n(\sqrt{0_R})$.

Pf: (1) We show $\Leftrightarrow$ for prime rings:

$\Leftarrow$: Suppose that $R \neq 0$ is not prime. Then $\exists\, I, J \trianglelefteq R$, s.t. $I \neq 0, J \neq 0$ but $I \cdot J = 0$. Then $\underbrace{M_n(I)}_{\text{ideals in } M_n(R) \to \text{see Ch. 1}} \cdot M_n(J) = 0$, thus $M_n(R)$ is not prime.

$\Rightarrow$: Suppose that $0 \neq M_n(R)$ is not prime. Then $\exists$ ideals $I', J' \trianglelefteq M_n(R)$ $I' \neq 0, J' \neq 0$ with $I' \cdot J' = 0$. But then $\exists$ ideals $I, J \trianglelefteq R$ with $I' = M_n(I)$ and $J' = M_n(J)$ and $I' \cdot J' = 0$ implies that $I \cdot J = 0$.
$\Rightarrow R$ is not prime. $\square$

(2) No proof.

§ 2.3. Uniform dimension (sometimes: = uniform rank) = Goldie dimension

Now approaching Goldie's thm!

Def 2.3.1 Let M be an R-module. A submodule $N \subset M$ is called essential (written $N \subseteq_e M$) if $N \cap N' \neq 0$ for every non-zero sub-module $N' \subset M$. In this case, M is called essential extension of N. A left ideal / right ideal / ideal $I \subseteq R$ is called essential left ideal / right ideal / ideal if it is an essential submodule.

dual notion to superfluous submodule!

Example 2.3.2
(1) Consider $(\mathbb{Q}, +)$ as a $\mathbb{Z}$-module. Then any two $\mathbb{Z}$-submodules of $\mathbb{Q}$ have nonzero intersection $\Rightarrow \mathbb{Z} \subseteq_e \mathbb{Q}$.
(2) Each nonzero submodule of $\mathbb{Z}/p^n\mathbb{Z}$, p prime, and $n \geq 1$, is essential.
(3) If K is a field and M a K-vector space, then M is the only essential submodule (subspace) of M.

Def A left ideal $I \subseteq R$ is called annihilator ideal if $I = \ell\text{-ann}_R(X)$ for some $X \subset M$, where M is an R-module.

$\{a \in R : a \cdot x = 0 \ \forall x \in X\}$

Lemma 2.3.3 (1) If R is a prime ring and $I \subset R$ is a nonzero ideal, then $I \subseteq_e {}_R R$ is essential.
(2) If I is a nilpotent ideal, then $\ell\text{-ann}_R(I) \subseteq_e R_R$ is essential.

Pf: (1) If $J \subset R$ is a nonzero left ideal, then $0 \neq I \cdot J \subset I \cap J$.
(2) Let $J \subset R$ be a nonzero right ideal. Then there is a $k \in \mathbb{N}$ with $JI^k \neq 0$ and $JI^{k+1} = 0$. Then $0 \neq JI^k \subset J \cap \ell\text{-ann}(I)$. □

Lemma 2.3.4 Let M be an R-module, $t \in \mathbb{N}_{\geq 1}$ and $N, N', P, M_1, N_1, \dots, M_t, N_t$ be submodules of M.
(1) If $N \subset P$, then $N \subseteq_e M$ is essential, if and only if, $N \subseteq_e P$ and $P \subseteq_e M$ are both essential.

(2) If $N_1 \subseteq_e M$ and $N_2 \subseteq_e M$, then $N_1 \cap N_2 \subseteq_e M$ is essential.

(3) If $N \subseteq_e M$ and $m \in M$, then $m^{-1}N = \{r \in R : rm \in N\} \subseteq_e {}_R R$. ← as left R-mod

(4) Let $P \subseteq_e N$ be essential. If $f : M \to N$ is an R-module homomorphism then $f^{-1}(P) \subseteq_e M$. In particular, for every $c \in N$, the left ideal $\{r \in R : rc \in P\} \subseteq_e R$ is essential.

(5) If $N_i \subseteq_e M_i$ $\forall i \in \{1, \dots, t\}$, then $N_1 \oplus \dots \oplus N_t \subseteq_e M_1 \oplus \dots \oplus M_t$.

(6) Let $N \subset M$ be a submodule. Then there is an $N' \subset M$, which is maximal with the property that $N \cap N' = 0$ and we have $N \oplus N' \subseteq_e M$.

(7) M is semi-simple if and only if M is the only essential submodule of M.

Pf: (1)+(2): use def.

(1) $\Rightarrow$: Let $N \subset_e M$. $N \subset P$

Show: $N \subset_e P$ and $P_e < M$

• Assume P not ess. in M: $\exists N' \leq M$ s.t. $P \cap N' = 0$ ($N' \neq 0$)

$\Rightarrow (P \cap N') \cap N = 0$

$\underbrace{(P \cap N)}_{N \subseteq P \Rightarrow = N} \cap N' = 0 \Rightarrow N \cap N' = 0$ i.e. N not essent. ↯

• N not ess in P $\Rightarrow \exists P' \leq P$ ($P' \neq 0$) s.t. $N \cap P' = 0$

$P' \leq M$ as well ↯ N ess.

$\Leftarrow$: ...

(2) Let $N' \leq M$ ($N' \neq 0$) $\Rightarrow N_1 \cap N' \neq 0$

$(N_1 \cap N') \cap N_2 \neq 0$ (N_2 essential)

(3) Let $I \leq R$ be a non-zero right ideal. If $Im = 0$, then $I \subset \{r \in R : rm \in N\}$.
If $Im \neq 0$ then $Im \cap N \neq 0 \Rightarrow I \cap \{r \in R : rm \in N\} \neq 0$.

(4) $P \leq_e N$, $f : M' \to N$. Let $M' \leq M$ be nonzero submodule. If $f(M') = 0$, then $M' \leq f^{-1}(P)$, so $f^{-1}(P) \cap M' \neq 0$.
If $f(M') \neq 0$, then $P \cap f(M') \neq 0$, since $P \leq_e N$. Hence $f^{-1}(P) \cap M' \neq 0$.
$\Rightarrow f^{-1}(P) \leq_e M$.
Let $c \in N$. Then $f : R \to N : \lambda \mapsto \lambda c$ is an R-module homomorphism
$\Rightarrow f^{-1}(P) = \{r \in R : rc \in P\} \subseteq_e R$ is essential.

(5) It is enough to consider $t = 2$. Look at $\pi_i : M_1 \oplus M_2 \to M_i$, $i = 1, 2$, and apply (4): $N_1 \oplus M_2$ and $M_1 \oplus N_2$ are essential in $M_1 \oplus M_2$.
Then use (2): $N_1 \oplus N_2 = (N_1 \oplus M_2) \cap (M_1 \oplus N_2) \leq_e M_1 \oplus M_2$.

(6) Set $\Omega := \{\tilde{N} \leq M : \tilde{N} \cap N = 0\}$ ($\Omega \neq \emptyset$ since $0 \in \Omega$). By Zorn's lemma, Ω has a maximal element N', and we claim that $N \oplus N' \leq_e M$ is essential.

Assume on the contrary, that $\exists\, Y \neq 0 \subset M$ with $Y \cap (N \oplus N') = 0$.
Then $N \cap (N' \oplus Y) = 0$.

$\Rightarrow Y \cap N = 0$ and $Y \cap N' = 0$
$N \cap N' = 0$ by def of Ω.

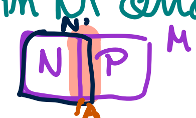
By maximality of $N' \Rightarrow Y = 0$ ↯.

(7) Let M be semi-simple and let $N \subsetneq M$ be nonzero submodule.
$\Rightarrow$: Then $\exists N' \subset M$ s.t. $M = N \oplus N'$, i.e. $N \cap N' = 0$ and so N is not essential.

$\Leftarrow$: Suppose that M has no proper essential submodule, and let $N \subset M$. By (6) $\exists N' \subset M$ s.t. $N \oplus N' \subseteq_e M$. Thus $N \oplus N' = M$ and M is s.s. (Charact. of s.s. of Ch I). □
Thm 1.12.4

Rmk 2.3.5 (1) By Lemma 2.3.4. (6) for any submodule N of M there is a submodule N' of M maximal w.r.t. the property $N \cap N' = 0$. This N' is called the complement to N in M. A submodule, which is a complement to some submodule of M, is called a complement submodule.

(2) If N is a complement submodule in M and $N \subsetneq N'$, then $\exists$ nonzero submodule $A \subset N'$ with $A \cap N = 0$.



$\hookrightarrow$ Pf: If N is a complement to $P \subseteq M$, then $A := N' \cap P$ has the desired property.

Def 2.3.6 An R-module M is called
uniform if $M \neq 0$ and each submodule of M is essential.
We say that M has finite uniform dimension if it contains no infinite direct sum of nonzero submodules.
$\hookrightarrow$ this is also called "rank of the module"

Examples: - M simple is uniform and has finite uniform dimension.
- If M is noetherian, then it is of finite uniform dimension.

Rmk 2.3.7 (1) If M has finite uniform dimension, then the same is true for any submodule of M.

(2) By Lemma 2.3.4. (6), M is uniform $\Leftrightarrow$ H contains no direct sum of nonzero submodules, ess. extension of M 6

(3) M is uniform $\Leftrightarrow$ Its injective hull is indecomposable [G-W, Lemma 5.14]
Moreover, M has finite uniform dimension if its inj. hull is a finite direct sum of uniform submodules.

Lemma 2.3.8 Let $M \neq 0$ be an R-module with finite uniform dimension.
(1) Then M has a uniform submodule.
(2) M contains an essential submodule which is a finite direct sum of uniform submodules.

Pf: (1) Assume on the contrary that M contains no essential submodule. Use remark 2.3.7. (2) to construct a sequence of nonzero submodules $(M_i)_{i\geq 0}$ s.t. $M \supset \bigoplus_{i\geq 0} M_i$. ↯ to M having finite uniform dimension: since M has no uniform submodule, M is not uniform. Set $M_0' := M$.
Let $k \geq 0$ and suppose that $M \supset (M_0 \oplus \dots \oplus M_{k-1} \oplus M_k')$ where all $M_i \neq 0$ are submodules of M and $M_k' \neq 0$ does not contain any uniform submodule. Thus $\exists$ nonzero submodules M_k, M_{k+1}' with $M_k' \supset M_k \oplus M_{k+1}'$ and M_{k+1}' does not contain a uniform submodule.
This gives us the required infinite sequence of submodules $(M_i)_{i\geq 0}$.
(2) By (1), M contains a uniform submodule. $\Rightarrow$ M contains a finite direct sum $M' = \bigoplus_{i=1}^{n} M_i$ of uniform submodules.
If M' is not essential, then there is a nonzero submodule $X \subset M$ with $M' \cap X = 0$. ← by Rmk 2.3.7. (1) Since X has finite uniform dimension, (1) implies that it contains a uniform submodule M_{n+1}.
$\Rightarrow M \supset \bigoplus_{i=1}^{n+1} M_i$.
Since M has finite uniform dimension, we obtain an essential submodule of M which is a finite direct sum of uniform sub=

modules. $\square$

Thm 2.3.9 Let M be a nonzero module with finite uniform dimension and let $\bigoplus_{i=1}^{n} U_i$ be a finite direct sum of uniform submodules which is essential in M.

(1) Any direct sum of nonzero submodules of M has at most n summands.

(2) A direct sum of uniform submodules of M is essential in M if and only if it has precisely n summands.

Pf: (1) Let $k \geq 1$ and $V_1, \ldots, V_k$ be nonzero submodules of M, s.t. $V_1 \oplus \ldots \oplus V_k \subseteq M$. Then $W := V_2 \oplus \ldots \oplus V_k \subseteq M$ is not essential (since $W \cap V_1 = 0$ and $V_1 \neq 0$.

W not essential

$\Rightarrow \exists\, i \in \{1, \ldots, n\}$ such that $W \cap U_i = 0$, w.l.o.g. $i = 1$.
Then $U_1 \oplus V_2 \oplus \ldots \oplus V_k$ is a direct sum. Repeating this process k times, it follows that $k \leq n$.

(2) $\Rightarrow$: A direct sum of k uniform submodules of M, which is not essential in M, has a complement N' by Lemma 2.3.4.(6) and N' has a uniform submodule by Lemma 2.3.8.
Thus we obtain a direct sum of $k+1$ uniform submodules in M.
This, together with (1), shows that any direct sum of n uniform summands is essential.

$\Leftarrow$: If $\bigoplus_{i=1}^{n'} U_i' \subseteq_e M$ is a direct sum of uniform submodules, then (1) implies $n' \leq n$. Exchanging the role of $\bigoplus_{i=1}^{n} U_i$ and $\bigoplus_{i=1}^{n'} U_i'$

shows that $n = n'$. $\square$

The nonzero integer n of thm 2.3.9 is called the <u>uniform dimension</u> (or <u>Goldie dimension</u>) of M, written $\mathrm{udim}(M) = n$.
We set $\mathrm{udim}(0) := 0$ and if M fails to have finite uniform dimension, then $\mathrm{udim}(M) = \infty$.

[McC-R, p 54, Ex 2.11]
<u>Example</u>: "Let k be a field and let $k[x,y]$ the commutative polyn. ring in 2 vars, and let $R := k[x,y]/(x,y)^n$. Then the sum $\sum_{i=0}^{n-1} k\bar{x}^i\bar{y}^{n-i-1}$ is a direct sum of uniform submodules of R, ($\bar{P}$ image of $P \in k[x,y]$ in R)
(e.g. $i=0$ $\quad k\bar{y}^{n-1} = M$ is uniform, since if $N' \subseteq M$, $N' \neq M$, some for $i = 0, \ldots, n-1$) $\quad y^{n-1} + (x,y)^n$, $\neq 0$

and further is essential in R. Thus $\mathrm{udim}(R) = n$.
(2) $R = k$ field, $M \cong k^n = \bigoplus_{i=1}^{n} k$. Each k is uniform k-mod, $\mathrm{udim}(M) = n$
(3) $R = k[x,y]/(xy)$. $M_1 = k[y] \cong R/x$ and $M_2 = k[x] \cong R/y$ are uniform

<u>Cor 2.3.10</u> Let M be an R-module.
(1) $\mathrm{udim}(M) = 1 \iff M$ is uniform.
$\mathrm{udim}(M) = 0 \iff M = 0$.
(2) Let $\mathrm{udim}(M) = n$ and $N \subset M$. Then $\mathrm{udim}(N) = n \iff N \subseteq_e M$ is essential.
(3) $\mathrm{udim}(M_1 \oplus M_2) = \mathrm{udim}(M_1) + \mathrm{udim}(M_2)$.
(4) M has finite uniform dimension $\iff M$ satisfies the ACC for complement submodules. Indeed, $\mathrm{udim}(M) =$ max. length of a chain of complement submodules.

<u>Pf</u>: (1) Use def.
$\Rightarrow$: $\mathrm{udim}(M) = 1 \Rightarrow \exists U_1 \in M$ uniform. by Lemma 2.3.8, U_1 essential $U_1 \subseteq_e M$
If $u \neq 0 \in M \Rightarrow U \cap U_1 \neq 0 \Rightarrow U$ ess. in M.
$\Leftarrow$: if M is uniform, then each subm. of M is essential, by Thm $\mathrm{udim} = 1$.
$\Rightarrow$: $\mathrm{udim}(M) = 0$: $\not\exists$ ess. submod^u in M, $M = 0$ (otherwise take $U = M$)
$\Leftarrow$: If $M = 0$, then by def. $\mathrm{udim}(M) = 0$.

(2) The property essential is transitive.
$\Rightarrow$: $\mathrm{udim}(N) = n$. If N not essential in M: $\exists U \subseteq M$ s.t. $U \cap N = 0 \Rightarrow U \oplus N \subseteq M$
$\mathrm{udim}(U \oplus N) \geq n+1$ ↯
$\Leftarrow$: If $N \subseteq_e M$, then $(\bigoplus U_i \cap N) \in N \Rightarrow \mathrm{udim}(N) \geq n$.
$\bigoplus U_i$, $\neq 0$

(3) Clear.