WEEK 11

Tuesday: Presentations on · Quaternions (Pau)
· Weyl algebras (Lucía)

Thursday: Continue with Section on semi-prime rings.

Recall: Idea is to generalize (commutative) reduced rings to noncommutative setting.
i.e. does not contain nilpotent elements

We defined, for ideal $I \subseteq R$
$$\sqrt{I} = \{s \in R : \text{if } T \text{ is an m-system with } s \in T, \text{ then } T \cap I \neq \emptyset\}$$
and showed that $\sqrt{I} = \bigcap_{\substack{I \subseteq \mathfrak{p} \\ \mathfrak{p} \subseteq R \text{ prime}}} \mathfrak{p}$ (Lemma 2.2.2)

We are in the middle of proving the following lemma (already shown: (a) $\Leftrightarrow$ (b).

Lemma 2.2.3 For an ideal $I \subseteq R$, TFAE:
(a) I is an intersection of prime ideals.
(b) If $x \in R$ with $xRx \subseteq I$, then $x \in I$.
(c) If $J \subseteq R$ is an ideal with $J^2 \subset I$, then $J \subseteq I$.
(d) If $J \subseteq R$ is a right ideal with $J^2 \subseteq I$, then $J \subseteq I$.
(e) If $J \subseteq R$ is a left ideal with $J^2 \subseteq I$, then $J \subseteq I$.
(f) $I = \sqrt{I}$.

Pf (a) $\Rightarrow$ (b) Suppose that $I = \bigcap_{\lambda \in \Lambda} \mathfrak{p}_\lambda$ with $\mathfrak{p}_\lambda$ in $\mathrm{Spec}(R)$ $\forall \lambda \in \Lambda$, and let $x \in R$ with $xRx \subseteq I$.
For all $\lambda \in \Lambda$, we have $xRx \subseteq \mathfrak{p}_\lambda \Rightarrow x \in \mathfrak{p}_\lambda$ by Lemma 2.1.2. $\Rightarrow$

$x \in \bigcap_{\lambda\in\Lambda} \mathfrak{p}_\lambda = I$.

(b)$\Rightarrow$(a) Let $x \in R\setminus I$. We have to show that $\exists\, \mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{p} \supset I$ and $x \notin \mathfrak{p}$.

Claim There is an m-system T with $x \in T \subset R\setminus I$.

If the Claim holds, then $I \in \Omega = \{J \subset R : I \subset J : J \cap T = \emptyset\}$. By Lemma 2.1.9, Ω has a maximal element $\mathfrak{p}$, which is a prime ideal and $x \notin \mathfrak{p}$.

Remains: Proof of Claim: We recursively define $T \subset R\setminus I$: Set $x_1 := x$. Let $n \in \mathbb{N}$ and $x_n \in T \subset R\setminus I$. Then $\exists\, r_n \in R$ with $x_n r_n x_n \in R\setminus I$ and we set $x_{n+1} := x_n r_n x_n$.

Set $T := \{x_n : n \in \mathbb{N}\}$. This is indeed an m-system: we have to verify that $\forall i,j \in \mathbb{N}_{\geq 1} : x_i R x_j \cap T \neq \emptyset$. Let $i,j \geq 1$. If $i \leq j$, then $x_{j+1} \in x_j R x_j \subset x_i R x_j$. If $i \geq j$, then $x_{i+1} \in x_i R x_i \subset x_i R x_j$. $\square_{\text{claim}}$

(b)$\Rightarrow$(d) For every $x \in J$, we have $xRx \subseteq J^2 \subset I$ and, and hence $x \in I$.

(d)$\Rightarrow$(c): Clear (ideal $\Rightarrow$ right ideal)

(c)$\Rightarrow$(b): Let $x \in R$ with $xRx \subseteq I$. Then $(RxR)^2 = (RxR)(RxR)$
$= R\underbrace{xRx}_{\subset I}R \subset \underset{I}{\underset{\shortparallel}{RIR}}$

and hence $x \in RxR \subset I$.

(b)$\Leftrightarrow$(e): by symmetry.

(f)$\Rightarrow$(a): By Lemma 2.2.2. I is an intersection of prime ideals.

(a)$\Rightarrow$(f): We have to show: $\sqrt{I} \subset I$ $\quad \left(I \subset \sqrt{I} \underset{2.2.2}{=} \bigcap_{\substack{I \subseteq \mathfrak{p} \\ \mathfrak{p} \in \operatorname{Spec}(R)}} \mathfrak{p}\right)$

Let $a \in R\setminus I$. By the Claim, there is an m-system T with $a \in T \subset R\setminus I$. Since $I \cap T = \emptyset$, the def. of $\sqrt{I}$ implies that $a \notin \sqrt{I}$. $\square$

Def 2.2.4

(1) An ideal satisfying the conditions of Lemma 2.2.3 is

called <u>semi-prime</u>. (Usually: $I \subseteq R$ semiprime :$\Leftrightarrow$ I is $\bigcap$ of prime ideals)

(2) A ring R is called semi-prime if 0 is a semi-prime ideal.

(3) $\sqrt{0}$ is called the <u>prime radical</u> or <u>nil radical</u> of R.

$\sqrt{0} \overset{L.2.2}{\subseteq} \{s \in R : \exists n ! \, s^n = 0\}$ $\leftarrow \sqrt{0} \subseteq \{\text{nilpotent elements of } R\}$

<u>Rmk</u>: • By convention $\bigcap_{\mathfrak{p} \in \emptyset} \mathfrak{p} = R$ (i.e. $\bigcap$ of empty family of p.i. $= R$)

so R itself is a semi-prime ideal. By def., every prime ideal is semi-prime.

• For every ideal I, we have $\sqrt{I} = \sqrt{\sqrt{I}}$ ($\because \sqrt{I} \overset{\text{Lemma 2.2.2}}{=} \bigcap_{\substack{I \subseteq \mathfrak{p} \\ \mathfrak{p} \in \operatorname{Spec}(R)}} \mathfrak{p}$

and $\sqrt{\sqrt{I}} = \bigcap_{\substack{\sqrt{I} \subseteq \mathfrak{p} \\ \mathfrak{p} \in \operatorname{Spec}(R)}} \mathfrak{p}$ some $\mathfrak{p}$'s !)

$\Rightarrow \sqrt{I}$ is semi-prime.

• If $R = 0$, then it has no prime ideals, and thus $\sqrt{0} = R$.

If $R \neq 0$, then it has at least one max. ideal, which is prime.

$\Rightarrow$ its prime radical $\sqrt{0}$ is a proper ideal of R, and by Lemma 2.2.2. $\sqrt{0} = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}$.

<u>Cor 2.2.5</u> Let $I \subseteq R$ be a semi-prime ideal. If $J \subseteq R$ is a right or left ideal and $J^n \subseteq I$ for some $n \geq 1$, then $J \subseteq I$.

<u>Pf</u>: Induction on n. For $n=1$, statement holds. Let $n \geq 2$ and suppose the claim holds for all powers $< n$. Since $2n-2 \geq n$, it follows that $(J^{n-1})^2 = J^{2n-2} \subset J^n \subset I$. Then $J^{n-1} \subset I$ by Lemma 2.2.3 (d) or (e): $J^2 \subset I \Rightarrow J \subset I$, so $J \subset I$ by induction hypothesis. $\square$

<u>Rmk</u> 2.2.6. (1) $\sqrt{0} \subset J(R)$, since every maximal ideal is prime

<u>ex</u>: • for $\sqrt{0} \subsetneq J(R)$: let $R = k[[x]]$ commutative, then $J(R) = \mathfrak{m} = \langle x \rangle$.

Since R is comm: Lemma 2.2.2: $\sqrt{0} = \{\text{nilpotent elements in } R\}$

$\Rightarrow$ Since $\mathbb{R}$ has no nilpotent elements: $\sqrt{0}=0 \Rightarrow \sqrt{0} \subsetneq J(R)$.

- If $R=\mathbb{R}[x_1,\dots,x_n]$ comm., then $J=\bigcap_{m\in \mathrm{Spec}(R)\,\max} m \subseteq \bigcap_{(a_1,\dots,a_n)\in\mathbb{R}^n} \langle x_1-a_1,\dots,x_n-a_n\rangle = 0$ (if $|R|=\infty$ or has enough pts)

$\Rightarrow \sqrt{0} = J(R) = 0$.

(2) Let $I\subseteq R$ be an ideal. Then I is semi-prime $\Leftrightarrow R/I$ is semi-prime.

(3) Since $\sqrt{0}$ is semi-prime, every nilpotent left ideal is contained in $\sqrt{0}$ by Cor. 2.2.5. By Lemma 2.2.2. $\sqrt{0}\subseteq\{\text{nilp. elts}\}$ but the inclusion can be strict.

<u>Cor 2.2.7</u>: TFAE: (a) R is semi-prime.
(b) $\sqrt{0}=0$.
(c) R has no nonzero nilpotent ideals.
(d) R has no nonzero nilpotent left ideal.

Pf: <u>(a) $\Leftrightarrow$ (b)</u>: Clear. ($\Rightarrow$ By Lemma 2.2.3: $0 \overset{(c)}{=} \sqrt{0}$. $\Leftarrow$: Lemma 2.2.3)
<u>(b) $\Rightarrow$ (d) $\Rightarrow$ (c)</u>: Follows from Rmk 2.2.6 (3).
<u>(c) $\Rightarrow$ (a)</u>: If there are no non-zero nilpotent ideals, then 0 satisfies Property (c) of Lemma 2.2.3. (If $J\subseteq R$ s.t. $J^2\subseteq 0$, then $J\subseteq 0$)
$\{a\in J: a^2=0\}=\{a\in J: a=0\}$ $\square$

To be continued after Christmas break...