WEEK 10

Recall: $\mathfrak{p}$ two-sided ideal $\subseteq R$ prime $\Leftrightarrow$ $\mathfrak{p} \subsetneq R$ and if $I, J \subseteq R$ s.t. $I \cdot J \subseteq \mathfrak{p}$ $\Rightarrow I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$
in $R/\mathfrak{p}$, the ideal 0 is prime (i.e. $R/\mathfrak{p}$ is prime ring)

§2.2. Semiprime rings = generalization of reduced commutative rings

Recall from commutative algebra

Prop: Let R be a commutative ring.

(a) $\bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p} = \{a \in R : a \text{ nilpotent, i.e. } \exists n \geq 1 \text{ s.t. } a^n = 0\}$.

(b) For any ideal $I \subseteq R$: $\bigcap_{\substack{\mathfrak{p} \in \mathrm{Spec}(R) \\ \mathfrak{p} \supseteq I}} \mathfrak{p} = \{a \in R : \exists n \geq 1 \text{ st. } a^n \in I\}$

this is $\sqrt{I}$ the RADICAL of I

(c) R does not contain any nilpotent elements $\Leftrightarrow 0 = \sqrt{0}$ i.e.
$0 = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$ is int. of p.i.'s

Pf: (a) $\supseteq$: if r is nilpotent and $\mathfrak{p} \subseteq R$ prime, look at $R/\mathfrak{p}$: since $R/\mathfrak{p}$ does not have any nilpotent elts. $\Rightarrow r + \mathfrak{p} = 0$ and $r \in \mathfrak{p}$.
$\subseteq$: Assume that $r \in R$ is not nilpotent, then $T := \{r^n : n \geq 1\}$ is an m-system. Then $T \cap (0) = \emptyset \underset{\text{Lemma 2.1.9 (2)}}{\Rightarrow} \exists$ p.i. $\mathfrak{p}$ with $\mathfrak{p} \cap T = \emptyset$ (i.e. $r \notin \mathfrak{p}$) $\Rightarrow r \notin \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$.

(b) follows from (a) by passing to the factor ring R/I.

(c) is a special case of (a).

We will see: semiprime ideals are noncomm. analogues of radical ideals, and semiprime rings $\leftrightarrow$ reduced rings in comm. alg.
I semiprime $= \bigcap \mathfrak{p}$ prime ids

Def 2.2.1 For an ideal $I \subseteq R$ set
$$\sqrt{I} = \{s \in R : \text{if } T \text{ is an } m\text{-system with } s \in T, \text{ then } T \cap I \neq \emptyset\}$$

Lemma 2.2.2 Let $I \subseteq R$ be an ideal.

(1) $\sqrt{I} \subset \{s \in R : s^n \in I$ for some $n \in \mathbb{N}\}$ and equality holds if R is commutative. ②

(2) $\sqrt{I} = \bigcap_{I \subset \mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$. In particular, $\sqrt{I}$ is an ideal in R.

Pf: (1) Since, for every $s \in R$, the set $\underbrace{\{s^n : n \in \mathbb{N}\}}_{=:T}$ is an m-system, the first inclusion follows.

this is mult. closed: $s^n \cdot s^m = s^{n+m} \in T$ and $T \cap I \neq \emptyset$ because $\exists n : s^n \in I$ by def.

Now suppose that R is commutative, let $s \in R$ and $n \in \mathbb{N}$ s.t. $s^n \in I$. Let T be an m-system with $s \in T$. Since R is commutative, $\exists r \in R$, s.t. $s^n r \in T$ ($\underbrace{srs}_{s^2 r} \in T$ etc, $\exists r$)

Since I is an ideal, also $s^n r \in I \Rightarrow T \cap I \neq \emptyset \Rightarrow s \in \sqrt{I}$.

(2) $\subseteq$: Let $s \in \sqrt{I}$. If $\mathfrak{p} \in \mathrm{Spec}(R)$ with $I \subseteq \mathfrak{p}$, then $T = R \setminus \mathfrak{p}$ is an m-system and $s \notin T$ ($\because$ otherwise: $(R\setminus\mathfrak{p}) \cap \mathfrak{p} \supset (R \setminus \mathfrak{p}) \cap I \neq \emptyset$ ↯) and $\Rightarrow s \in \mathfrak{p}$.

$\supseteq$: Suppose that $s \notin \sqrt{I}$. Then there is an m-system T with $s \in T$ and $I \cap T = \emptyset$. Thus $I \in \Omega = \{J \subset R : I \subset J, J \cap I = \emptyset\}$ and by Lemma 2.1.9 there is a maximal element $\mathfrak{p} \in \Omega$. Since $\mathfrak{p} \in \mathrm{Spec}(R)$ with $I \subseteq \mathfrak{p}$ (prime) and $s \notin \mathfrak{p}$ the assertion follows. $\square$

<u>Lemma 2.2.3</u> For an ideal $I \subseteq R$, TFAE:

(a) I is an intersection of prime ideals.

(b) If $x \in R$ with $xRx \subseteq I$, then $x \in I$.

(c) If $J \subseteq R$ is an ideal with $J^2 \subset I$, then $J \subseteq I$.

(d) If $J \subseteq R$ is a right ideal with $J^2 \subseteq I$, then $J \subseteq I$.

(e) If $J \subseteq R$ is a left ideal with $J^2 \subseteq I$, then $J \subseteq I$.

(f) $I = \sqrt{I}$.

Pf <u>(a) $\Rightarrow$ (b)</u> Suppose that $I = \bigcap_{\lambda \in \Lambda} \mathfrak{p}_\lambda$ with $\mathfrak{p}_\lambda$ in $\mathrm{Spec}(R)$ $\forall \lambda \in \Lambda$, and let $x \in R$ with $xRx \subseteq I$.

For all $\lambda \in \Lambda$, we have $xRx \subseteq \mathfrak{p}_\lambda \Rightarrow x \in \mathfrak{p}_\lambda$ by Lemma 2.1.2. $\Rightarrow$

$x \in \bigcap_{\lambda\in\Lambda} \mathfrak{p}_\lambda = I$.

<u>(b) $\Rightarrow$ (a)</u> Let $x \in R\setminus I$. We have to show that $\exists\, \mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathfrak{p} \supset I$ and $x \notin \mathfrak{p}$.

<u>Claim</u> There is an m-system T with $x \in T \subset R\setminus I$.

If the Claim holds, then $I \in \Omega = \{J \subset R : I \subset J : J \cap T = \emptyset\}$. By Lemma 2.1.9, Ω has a maximal element $\mathfrak{p}$, which is a prime ideal and $x \notin \mathfrak{p}$.

Remains: <u>Proof of Claim</u>: We recursively define $T \subset R\setminus I$: Set $x_1 := x$. Let $n \in \mathbb{N}$ and $x_n \in T \subset R\setminus I$. Then $\exists\, r_n \in R$ with $x_n r_n x_n \in R\setminus I$ and we set $x_{n+1} := x_n r_n x_n$.

Set $T := \{x_n : n \in \mathbb{N}\}$. This is indeed an m-system: we have to verify that $\forall i,j \in \mathbb{N}_{\geq 1} : x_i R x_j \cap T \neq \emptyset$. Let $i,j \geq 1$. If $i \leq j$, then $x_{j+1} \in x_j R x_j \subset x_i R x_j$. If $i \geq j$, then $x_{i+1} \in x_i R x_i \subset x_i R x_j$. $\square_{\text{claim}}$

Rest of the week: Presentations

<u>Tuesday</u> 3/12/24: Daniel Heriatakis: Coefficient quivers of Knörrer invariant algebras

<u>Thursday</u> 5/12/24: Markus Tripp: Group algebras

Manuel Timmel: Clifford algebras