WEEK 3

<u>Recall</u>: Composition series + module of f. length.

<u>Lemma 1.2.11</u>: Let M be an R-module. Then each two finite chains of submodules have equivalent refinements.
Pf: Omitted [G-W, Thm 4.10 "Schreier's thm"]

<u>Theorem 1.2.12</u> (Jordan-Hölder) Let M be an R-module. If M has finite length, then any two composition series are equivalent. In part. all composition series have the same length. ie. each chain (*) can be refined to ~~a~~ comp. series

Pf: [See G-W, Thm 4.11] Sketch:
If M has 2 decomp. series $M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_e = 0$
$M = N_0 \supsetneq N_1 \supsetneq \dots \supsetneq N_k = 0$
then by 1.2.11 $\exists$ equiv. refinement $M = M_0' \supseteq \dots \supseteq M_m' = 0$
$M = N_0' \supseteq \dots \supseteq N_m' = 0$
and $\exists \sigma \in S_m$ s.t. $M_{i-1}'/M_i' \cong N'_{\sigma(i)-1}/N'_{\sigma(i)}$. Then, because M_{i-1}/M_i is simple, the quotients M_{i-1}'/M_i' are in order, but with possible repetitions. Now argument for permutation (!) $\square$

<u>Thm 1.2.13</u> Let M be an R-module.
(1) TFAE: (a) M has a composition series.
(b) $\ell_R(M) < \infty$.
(c) M is noetherian and artinian.
(2) If $0 \to L \to M \to N \to 0$ is a s.e.s., then $\ell(L) + \ell(N) = \ell(M)$.

<u>Pf</u>: (1) (a)$\Leftrightarrow$(b): Jordan-Hölder
[Sketch] (a) $\Rightarrow$ (c): DCC ✓ if DCC is satisfied, then also ACC.

(2) We show special case $L \subseteq M \Rightarrow 0 \to L \hookrightarrow M \to M/L \to 0$.
If $L=0$ or $L=M$, then ✓.
So assume $0 \subsetneq L \subsetneq M$. [G-W, 4, 12]
Choose composition series:

$$L_m = 0 \subsetneq L_{m-1} \subsetneq \dots \subsetneq L_0 = L$$
$$M_n/L = 0 \subsetneq M_{n-1}/L \subsetneq \dots \subsetneq M_0/L = M/L \qquad \text{for } L \text{ and } M/L.$$

Since the chain $L_m = 0 \subsetneq \dots \subsetneq \underset{L}{\underset{\shortparallel}{L_0}} \subsetneq M_{n-1} \subsetneq \dots \subsetneq M_0 = M$ (M_{n-1}/L_0 simple ← part of comp. series of M/L.)
is a composition series for M, the result follows. □

We end this section with some important facts from module theory that we will not prove:

Rmk 1.2.14

(1) A ring R is said to be <u>local</u> if the set of non-invertible elements is an ideal.
<u>Ex</u>: $R = k$ a field, or $R = k[[x]]$, $k[x]_{(x)}$ commutative, then $\mathfrak{m}$: {non-invertible elements} $= \langle x \rangle$ is unique max. ideal

(2) If M is an indecomposable module of finite length, then $\mathrm{End}_R M$ is local and the non-invertible elements are nilpotent.
<u>Ex</u>: $R = M_2(k)$, then $\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$ is simple $(Re_1 / J \cdot R) = S_1$ (J a Jacobson rad → see later)
$\mathrm{End}_R(S_1) = \{ \varphi : \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{\cdot A} \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \}$
A is of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ (otherwise $\mathrm{im}\, A \not\subseteq S_1$)
$(A \text{ inv} \Leftrightarrow ad \neq 0) \Rightarrow (A \text{ not inv.} \Leftrightarrow A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix})$

Thm 1.2.15

Let M be a noetherian module.
(1) There are indec. submodules $M_1, \dots, M_r$ of M s.t. $M = M_1 \oplus \dots \oplus M_r$.
(2) (Thm of Krull-Remak-Schmidt-Azumaya aka "Krull-Schmidt")

If M is artinian (i.e. M is of finite length), then the decomp. ③
of (1) is unique up to reordering the summands:
If $M = N_1 \oplus \dots \oplus N_s$, then $r = s$ and $\exists \sigma \in S_r$ s.t. $M_i \cong N_{\sigma(i)}$
$\forall i \in \{1, \dots, r\}$.

(2) can be stated for objects M in an additive cat. of

<u>Rmk 1.2.16</u> (2) is important in comm. algebra (CM-represent. theory), in part. Krull-Schmidt property for <u>noetherian</u> (f.g.) modules M holds if base ring R is Henselian.

KRS does not hold for $R = k[x,y]$! Ex: $m = \langle x, y\rangle$ $n = \langle x-1, y\rangle$
can show: $m \oplus n \cong R \oplus (m \cap n)$
but neither m, n is $\cong R$!

See Book [Leuschke-Wiegand, Chapter 1] for more on KRS.

1.3. Simple, semisimple, and artinian rings

<u>Def</u> 1.3.1 A ring R is called
- <u>simple</u> if $R \neq 0$ and 0 and R are the only two-sided ideals
- <u>semi-simple</u> if R is a semi-simple left-module.

<u>Ex</u>: • $R = D$ division ring is simple
• $R = \mathbb{Z}/n\mathbb{Z}$ is simple $\Leftrightarrow$ n is prime
is s.s. $\Leftrightarrow$ n is squarefree
• $R = \mathbb{C}\langle x, \partial_x\rangle / \langle \partial_x x - x\partial_x - 1\rangle$ WEYL-ALGEBRA is simple

Idea: any diff. op is of the form $p_n(x)\partial_x^n + p_{n-1}\partial_x^{n-1} + \dots$ With relation,
Induct.: If $p(x) \in I$, then $1 \in I$.
If $\partial_x \in I$, then $1 \in I$

<u>Rmk 1.3.2</u>
(1) A simple ring does not need to be a simple left R-module and it doesn't need to be semisimple.
<u>Ex</u>: Weyl Algebra A_1 has left ideals: e.g. $\langle \partial_x \rangle$
$A_1 \cdot \partial_x$

(2) Let R_1 and R_2 be semi-simple rings. Then their product $R_1 \times R_2$ is a semi-simple ring. $\Rightarrow$ Finite direct products of semi-simple rings are semi-simple.

Pf (of (2)) As $(R_1 \times R_2)$-module, we have:
$$(R_1 \times R_2) = (R_1 \times \{0\}) \times (\{0\} \times R_2).$$
The $(R_1 \times R_2)$-submodules of $(R_1 \times \{0\})$ are precisely of the form $M \times \{0\}$, where M is an R_1-submodule of R_1. $\square$

$ab = 0 \Rightarrow a = 0$ or $b = 0$

(3) Let R be a domain, but not a division ring. Then R has no simple submodules, in part., R is not semi-simple.

Pf (of (3)): Choose a $0 \neq I \subsetneq R$ submod. (= left ideal of R). Let $a \in I$, $a \neq 0$. Then $0 \neq Ra \subseteq I$. If $Ra \neq I$, then I is not simple and we are done. If $Ra = I$, then consider $Ra^2 = (Ra)a \subseteq I$. If Ra^2 were $= Ra$ ($= I$), then there would be a $b \in R$ s.t. $a = ba^2 \Rightarrow (ba - 1)a = 0$. Since R is a domain and $a \neq 0$, we must have $ba = 1$. Thus $\forall c \in R : c = cba \in Ra$
$\Rightarrow Ra \supseteq R$ ↯ to $Ra = I \subsetneq R$.

(4) The R-submodules of R are precisely the left ideals of R (by def.). Thus, by Thm 1.2.4., R is s.-s. $\Leftrightarrow$ every left ideal is a direct summand of R.

Thm 1.3.3 TFAE:
(a) Every R-module is semi-simple.
(b) R is a semi-simple ring.
(c) R is a direct sum of finitely many minimal left ideals.

In part., if R is semi-simple, then R is left artinian.

Pf: (a) $\Rightarrow$ (b): Clear (def!)

(b) ⇒ (c) Since simple submodules are precisely the min. left ideals of R, we write $R = \bigoplus_{i \in I} L_i$ ← min. left ids

def of $\oplus$: any elt. is finite sum!

We have to show that this sum is finite: There is a finite $J \subset I$ such that $1 = \sum_{j \in J} x_j$ for $x_j \neq 0 \in L_j$. Then $\forall a \in R: a = a \cdot 1 = \sum_{j \in J} a x_j \subset \sum_{j \in J} L_j$

$\Rightarrow R \subset \sum_{j \in J} L_j \Rightarrow R = \bigoplus_{j \in J} L_j$.

(c) ⇒ (b): Clear.

(b) ⇒ (a): Every free R-module is s.-s. (since it is a $\oplus$ of simples of R). Since any module M can be written as a factor module of a free module, it is s.-s. (Cor 1.2.5).

For the last statement, if R is s.-s., then $R = \bigoplus_{i=1}^{t} L_i$. So R has a composition series $0 \subsetneq L_1 \subsetneq L_1 \oplus L_2 \subsetneq \dots \subsetneq L_1 \oplus \dots \oplus L_t$, so it is of fi. length and hence left artinian. $\square$

Cor. 1.3.4. Let R be semi-simple. Then there are only finitely many isomorphism classes of simple R-modules.

Pf: By the Thm, we can write $R = \bigoplus_{i=1}^{s} L_i$, where L_i are minimal left ideals of R. Let N be a simple R-module. Choose $x \neq 0 \in N$ and consider the epimorphism → because Rmk 1.2.2 (1): $N = Rx$

$$\varphi: R \xrightarrow{\cdot x} N : r \mapsto r \cdot x$$

$\Rightarrow \exists i \in \{1, \dots, s\}$ s.t. $\varphi(L_i) \neq 0$. Then $\varphi|_{L_i} : L_i \to N$ is a $\neq 0$ homomorphism $\underset{\text{Schur's lemma}}{\Rightarrow} L_i \cong N$. $\Rightarrow N \in \underbrace{\{L_1, \dots, L_s\}}_{\text{the } L_i \text{ are the isom. classes}}$. $\square$

Ex. 1.3.5 Let D be a division ring and $n \geq 1$. Then $M_n(D)$ is simple and semi-simple with centre $Z(M_n(D)) = \underbrace{Z(D) \cdot \mathbb{1}_n}_{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \lambda \in D}$.

Pf: [Sketch] 3 Steps:

(1) Let $0 \neq I \subseteq M_n(D)$ be a two sided ideal. We show $I = M_n(D)$.

Let E_{ij} be the matrix: $\begin{array}{c|c} & j \\ i & \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$.

Since $I \neq 0$, $\exists\, A = (a_{ij})_{i,j \in \{1,\dots,n\}}$ with $A \in I$ and some indices k, ℓ, s.t. $a_{k\ell} \neq 0$. Now for all $i \in \{1,\dots,n\}$, set

$$B_i = E_{ik} \underbrace{A E_{\ell i}}_{\begin{bmatrix} 0 & a_{1\ell} \\ & \vdots & 0 \\ & a_{n\ell} \end{bmatrix}}$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$

i-th column

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} E_{ij} = \begin{cases} 11: & \text{1st col} \\ 21: & \begin{bmatrix} a_{12} & 0 & 0 \\ a_{22} & 0 & 0 \\ a_{32} & 0 & 0 \end{bmatrix} \\ 12: & \begin{bmatrix} 0 & a_{11} & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & 0 \end{bmatrix} \end{cases}$$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ - & a_{k\ell} & - \\ 0 & 1 & 0 \end{bmatrix}$

$\rightsquigarrow B_{ii} = a_{k\ell}$ and all other entries 0, i.e. $B_i = a_{k\ell} E_{ii}$.

Since $B_i \in I$ $\forall i$ (I two sided ideal!), we must also have

$$B = B_1 + \dots + B_n \in I.$$

But $B = \underset{\neq 0}{a_{k\ell}} \cdot \mathbb{1}_n$ is invertible ($B^{-1} = a_{k\ell}^{-1} \mathbb{1}_n$) $\Rightarrow I = M_n(D)$.

(2) Check that $L_i = \left[\begin{array}{c|c|c} 0 & \begin{matrix} * \\ \vdots \\ * \end{matrix} & 0 \end{array}\right]$ are all minimal left ideals of $M_n(D)$. [$L_i = M_n(D) e_i$]

i-th column

$\Rightarrow M_n(D) = L_1 \oplus \dots \oplus L_n$. Thus $M_n(D)$ is s.-s. by Thm 1.3.3.

(3) Claim about centre (!) Take elts in centre: $A \in C(M_n(D))$

- $\Rightarrow \underbrace{A E_i}_{\text{i-th col}} = \underbrace{E_i A}_{\text{i-th row}} \Rightarrow A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$
- $\underbrace{A E_{12}}_{\begin{bmatrix} 0 & \lambda_1 & 0 \\ 0 & \dots & \end{bmatrix}} = \underbrace{E_{12} A}_{\begin{bmatrix} 0 & \lambda_2 & 0 \end{bmatrix}}$ etc $\Rightarrow \lambda_1 = \dots = \lambda_n = \lambda$
- Now: $\underbrace{A \cdot \begin{bmatrix} \mu & & 0 \\ & \ddots & \\ 0 & & \mu \end{bmatrix}}_{\begin{bmatrix} \lambda\mu & & 0 \\ & \ddots & \\ 0 & & \lambda\mu \end{bmatrix}} = \begin{bmatrix} \mu & & 0 \\ & \ddots & \\ 0 & & \mu \end{bmatrix} A \quad \forall \mu \in D$

$= \begin{bmatrix} \mu\lambda & & 0 \\ & \ddots & \\ 0 & & \mu\lambda \end{bmatrix} \Rightarrow \lambda \in Z(D)$. $\square$

<u>Thm 1.3.6</u> The ring R is semi-simple $\iff$ every left ideal L is of the form $L = Re$ for some idempotent $e \in R$. (e idempotent: $e^2 = e$) always: 0, 1 idempotent

<u>Pf</u>: $\Rightarrow$: Let R be semi-simple and let $L \subset R$ be a left ideal.
Then (Thm 1.2.4 (c)) $\exists L'$ left ideal with $R = L \oplus L'$.
Thus $1 = x + y$ with $x \in L$ and $y \in L'$. Then

$$\underline{x} = x \cdot 1 = x(x+y) = \underline{x^2} + xy \Rightarrow xy = \overset{\in L}{x} - \overset{\in L}{x^2} \in L$$

Since L' is a left ideal and $y \in L'$, we have $xy \in L'$.
$\overset{L \cap L' = 0}{\Rightarrow}$ $xy = 0$, and thus $x = x^2 + 0$ is idempotent.
We have $Rx \subset L$ and $\forall a \in L$ we compute

$$a = a \cdot 1 = ax + ay \rightsquigarrow \underbrace{a - ax}_{\in L} = \underbrace{ay}_{\in L'} \in L \cap L' = 0$$

So: $a = ax$, i.e. $L \subset Rx$. ✓
$\Leftarrow$: Suppose, every left ideal is of the form $L = Re$, with $e = e^2 \in R$.
We show that every left ideal is $\oplus$ of $R \underset{\text{Thm 1.2.4 (c)}}{\Rightarrow} R$ s.-s.

Set $L' := R(1-e)$.
<u>Claim</u>: $R = Re \oplus R(1-e)$.
$\supseteq$: clear
$\subseteq$: Note that L' is left ideal of R and $\underset{(1-e)(1-e)}{(1-e)^2} = 1 - e - e + e^2 = 1 - 2e + e = 1 - e$

$\forall x \in R$: $x = xe + x(1-e)$ (clear!) $\Rightarrow R = Re + R(1-e)$.
Assume that $x \in Re \cap R(1-e)$: then $\exists s, r \in R$, s.t.

$$x = re = s(1-e).$$

$\Rightarrow xe = (re) \cdot e = re^2 = re = x$ and $xe = s(1-e)e = s\underbrace{(e - e^2)}_{0} = 0$
$\Rightarrow x = 0$ and $Re \cap R(1-e) = 0$.
Hence $R = Re \oplus R(1-e)$. $\square$

<u>Thm 1.3.7</u> (1) Let $R = L_1 \oplus \ldots \oplus L_s$ be a direct sum of minimal

left ideals. Then $\exists e_1,\dots,e_s \in R$ s.t.:

(a) $e_1,\dots,e_s$ are nonzero idempotents,

(b) For any $i,j \in \{1,\dots,s\}$, $i \neq j$, we have $e_ie_j = 0$.

(c) $1 = e_1 + \dots + e_s$

(d) Any e_i, $i \in \{1,\dots,s\}$, cannot be written as $e_i = e_i' + e_i''$ with e_i', e_i'' nonzero and $e_i'e_i'' = 0$.

(2) Let R be s.-s. and let $e_1,\dots,e_s$ be elements satisfying (1)(a)-(d). Then the left ideals $Re_1,\dots,Re_s$ are minimal and $R = \bigoplus_{i=1}^{s} Re_i$.

Pf: (1) For $i \in \{1,\dots,s\}$, let $e_i \in L_i$ s.t. $\underbrace{1 = e_1 + \dots + e_s}_{}\overset{(c)}{}$. With the same argument as in Pf of Thm 1.3.6, we can show that $L_i = Re_i$ and e_i idempotent $\forall i \in \{1,\dots,s\}$. $\Rightarrow$ (a)

Then $e_1 = e_1(e_1 + \dots + e_s) = \overset{e_1}{\overbrace{e_1^2}} + \sum_{i=2}^{s} e_1e_i \Rightarrow 0 = \sum_{i=2}^{s} \overset{0}{e_1} e_i \in \overset{R}{\bigoplus_{i=2}^{s}} L_i$.

$\Rightarrow e_1e_i = 0$ (if $\oplus = 0 \Rightarrow$ all factors $= 0$).

Similarly, show that $e_je_i = 0$ $\forall i,j \in \{1,\dots,s\}$ distinct. $\Rightarrow$ (b)

For (d), assume on the contrary, $\exists i$ s.t. $e_i = e_i' + e_i''$ and $e_i'e_i'' = 0$
$\neq 0$ idempotents

But then $L_i = Re_i' \oplus Re_i''$ ↯ to minimality of L_i. $\Rightarrow$ (d)

(2) $\forall i \in \{1,\dots,s\}$, set $L_i = Re_i$. We have to show that L_i is minimal left ideal. Assume on the contrary, that L_i is not minimal: Then $\exists J_i \subsetneq L_i$, $0 \neq$. Since R is semi-simple, L_i is semi-simple (Thm 1.2.4)

$\Rightarrow \exists J_i'$ s.t. $L_i = J_i \oplus J_i'$. Now set $e_i = e_i' + e_i''$. ↯ to (1)(d).

From (1)(c) we get $R = Re_1 + \dots + Re_s$, so just remains to show $\oplus$.

Let $j \in \{1,\dots,s\}$ and $x_j \in L_j \cap \left(\sum_{i \neq j} L_i\right)$. Then $x_j = r_je_j = \sum_{i \neq j} r_ie_i$

$\overset{\cdot e_j}{\Rightarrow} x_j \cdot e_j = \underbrace{r_je_je_j}_{= r_je_j = x_j} = \sum_{i \neq j} r_i \underbrace{e_ie_j}_{=0}$ $\Big\}$ $x_j = 0$, hence $\oplus$. □

Def A family $(e_1,\dots,e_s)$ satisfying properties (1)(a)-(c) is called a <u>complete system</u> <u>of orthogonal idempotents</u>. If, in addition,

(d) holds, then it is a primitive system of idempotents.