WEEK 2

<u>Recall</u>: last week noetherian, artinian modules + free modules

Now we will define simple + semi-simple modules, in order to study the structure of artinian + (semi-)simple rings

<u>Example 1.1.9</u>

(1) Consider $\Lambda = M_2(R)$, R any ring.

Then $I_1 := \Lambda \underset{\substack{\uparrow \\ \text{idempotent:} \\ \text{see later}}}{e_1} = \begin{bmatrix} R & R \\ R & R \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & 0 \\ R & 0 \end{bmatrix} = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, a, b \in R \right\}$ is a left ideal.

Similar: $I_2 := \Lambda e_2 = \begin{bmatrix} R & R \\ R & R \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & R \\ 0 & R \end{bmatrix}$ is a left ideal (but not right!)

$\rightsquigarrow$ "$\Lambda = I_1 \oplus I_2$"

For the other side: $e_1 \Lambda = \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix}$ and $e_2 \Lambda = \begin{bmatrix} 0 & 0 \\ R & R \end{bmatrix}$ are right ideals

(2) Let $n \in \mathbb{N}$. If $I \subseteq R$ is a left ideal, then $M_n(I) \subseteq M_n(R)$ is a left ideal. [Pf: exercise: $n=2$ $\begin{bmatrix} I & I \\ I & I \end{bmatrix} \subset \begin{bmatrix} R & R \\ R & R \end{bmatrix}$ check axioms.]

(3) Let $n \in \mathbb{N}$. Then R is left noetherian $\iff M_n(R)$ is left noetherian.

<u>Pf(Sketch)</u>: $\Rightarrow$: as (left) R-module: $M_n(R) \cong R^{n^2}$.

$\Leftarrow$: If $(I_j)_{j \in J}$ is an ascending chain of ideals in R, then $M_n(I_j)_{j \in J}$ is ascending chain of ideals in $M_n(R)$ $\xRightarrow{M_n(R) \text{ has ACC}}$ R has ACC. □

<u>Note</u>: R and $M_n(R)$ have "the same module category" $\rightsquigarrow$ they are Morita equivalent.

(4) The ring $\begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix} \subset M_2(\mathbb{Q})$ is right noetherian, but not left noetherian (${}_{\mathbb{Z}}\mathbb{Q}$ is not noetherian!) asc. chain of submods e.g. $\langle \tfrac{1}{2} \rangle \subsetneq \langle \tfrac{1}{2}, \tfrac{1}{4} \rangle \subsetneq \cdots$

The ring $\begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix} \subset M_2(\mathbb{R})$ is right artinian, but not left artinian.

(5) Let M be left noetherian and $f: M \to M$ surjective. Then f

is an isomorphism.

Pf: Consider $f^n(x) = \underbrace{f \circ \dots \circ f}_{n}(x)$: For $n \in \mathbb{N}$, set
show f inj!

$$M_n = \ker(f^n) = \{x \in M : f^n(x) = 0\}.$$

Then if $x \in M_i \Rightarrow f^i(x) = 0 \Rightarrow \underset{\forall k \geq 0:}{} f^{i+k} = f^k(f^i(x)) = f^k(0) = 0.$

$\Rightarrow M_1 \subseteq M_2 \subseteq \dots$ is AC $\overset{M \text{ noeth}}{\Longrightarrow} \exists N \in \mathbb{N}$ s.t. $M_N = M_{N+1}$.

Let now $x \in M_1$. Since f is surjective: $M = f^N(M)$, so there is a $y \in M$, s.t. $f^N(y) = x$.

But then $\underset{x \in M_1}{0 =} f(x) = f(f^N(y)) = f^{N+1}(y) \Rightarrow y \in M_{N+1} = M_N$.

$\Rightarrow f^N(y) = 0 \Rightarrow x = 0. \Rightarrow f$ is injective. □

(6) Let R be left noetherian and $a, b \in R$ s.t. $ab = 1$. Then $ba = 1$.

Pf: We have $R = Rab \subset Rb \subset R \Rightarrow R = Rb$.

Thus the homomorph. $f: R \to R : x \mapsto xb$ is surjective and by (5) it is an isomorphism.

Now compute: $f(1 - ba) = (1 - ba)b = b - b\underbrace{ab}_{1} = b - b = 0$

$\underset{f \text{ inj.}}{\Longrightarrow} 1 - ba = 0 \Leftrightarrow 1 = ba.$ □

§ 1.2. Simple + semi-simple modules

Def 1.2.1. An R-module M is said to be *simple* if $M \neq 0$ and 0 and M are its only submodules.
M is *semi-simple* if it is a direct sum of simple modules.
(Thm 1.2.4: any sum of simple mods is s.s.!)
The 0-module is def. to be semi-simple (empty direct sum).

Ex: (1) If R is any ring, then R is simple as a module over itself $\Leftrightarrow$ R is a division ring (see Lemma 1.1.1)
only left ideals are 0 and R
Any module M over a division ring is free, so every M is a semi-simple module. (Lemma 1.1.2)
(2) The simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$ for p prime.
(3) k^n considered as a column vector becomes a simple left $M_n(k)$-module. Can be seen with Morita theory (or by matrix mult!)

Remark 1.2.2
(1) If M is simple, then $M = Rx$ for every $x \neq 0$ in M.
Let $x \neq 0 \in M \Rightarrow Rx \subseteq M \overset{M \text{ simple}}{\Rightarrow} Rx = M$
(2) (a) $\forall x \in M$, the map $\varphi : R/\mathrm{ann}_R(x) \to Rx : r + \mathrm{ann}_R(x) \mapsto rx$ is an R-module isom. (!) isom thm: $R \xrightarrow{\cdot x} Rx \to 0$, $\mathrm{ann}_R(x) = \ker(\cdot x)$)
(b) M is cyclic $\Leftrightarrow M \cong R/L$ for some left ideal L in R.
($\Leftarrow$: generator: $1+L$, $\Rightarrow$ If $M = {}_R\langle a \rangle$, then: $R \xrightarrow{\cdot a} M \to 0 \Rightarrow M \cong R/\ker(\cdot a)$) ($\supseteq \ker(\cdot a)$)
(c) M is simple $\Leftrightarrow M \cong R/L$ for some max. left ideal $L \subset R$. ← needed often!!
Have bij: ideals in R/L and ideals in R cont. L.

(3) (a) Every ideal is contained in a maximal ideal. (Special case of 3(b))

(b) Let M be a f.g. R-module. Then every proper submodule $N \subsetneq M$ is contained in a max. submodule $N^* \subset M$ (Idea: Use Zorn's lemma for M/N).

The $\mathbb{Z}$-module $\mathbb{Q}$ has no max. submodules.

<u>Lemma 1.2.3</u>: Let M be an R-module, $N \subset M$ a submodule, and $(M_i)_{i\in I}$ a family of simple submodules s.t. $M = N + \sum_{i\in I} M_i$. Then $\exists$ a subset $J \subset I$ s.t. $M = N \oplus \bigoplus_{j\in J} M_j$.

<u>Proof</u>: (needs Zorn's lemma!)

Consider
$$\Omega := \{ I' \subset I : \text{the sum } N + \sum_{j\in I'} M_j \text{ is direct} \}.$$

Since $\emptyset \in \Omega$, we have that $\Omega \neq \emptyset$.

First proof the

<u>Claim A</u> Ω is inductively ordered (w.r.t. set theoretical inclusion)

i.e. every chain of elts in Ω has upper bound

$\hat{=}$ tot. ordered subsets

<u>Pf of A</u>: Let $Y \subset \Omega$ be a non-empty totally ordered subset of Ω. Let $I_0 = \bigcup_{I' \subset Y} I'$, then, $I_0 \in \Omega$. This implies that I_0 is an

(we will show!)

I'_1, I'_2 ...

upper bound for Y in Ω, and Claim A follows.

Now show that $I_0 \in \Omega$: have to prove that $N + \sum_{j\in I_0} M_j$ is $\oplus$.

Recall that a sum of submodules $\sum_{s\in S} L_s$ of some R-modules is direct $\iff$ for every <u>finite</u> $T \subset S$ the sum $\sum_{t\in T} L_t$ is direct. Thus: sufficient to show that $N + \sum_{j\in I_1} M_j$ is direct for every finite $I_1 \subset I_0$.

Since Y is totally ordered, for every such I_1 there is an $I' \in Y$ s.t. $I_1 \subset I'$. By def. of Ω, the sum $N + \sum_{j\in I'} M_j$ is direct, and hence also the smaller sum $N + \sum_{j\in I_1} M_j$ is $\oplus$. $\square$ (for Claim A).

Now by Zorn's lemma $\exists$ max. element $J \in \Omega$.

<u>Claim B</u>: $M = N \oplus \bigoplus_{j \in J} M_j$.

<u>Pf B</u>: By def. the sum is direct (and $\subseteq N + \sum_{i \in I} M_i$ $\overset{\shortparallel}{M}$ $N+\sum_{j \in J} M_j$), so we only have to show:

$$M = N + \sum_{i \in I} M_i \subseteq N \oplus \bigoplus_{j \in J} M_j.$$

Assume on the contrary, that there is an $i \in I$ with $M_i \not\subseteq N + \sum_{j \in J} M_j$

Then $i \notin J$ and $\underbrace{M_i \cap (N \oplus \bigoplus_{j \in J} M_j)}_{(*)} \subsetneq M_i$.

Since by assumption M_i is simple, $(*)$ is 0, and thus the sum

$$N \oplus \bigoplus_{j \in J} M_j + M_i \text{ is direct.}$$

$$N \overset{\shortparallel}{+} \sum_{j \in J \cup \{i\}} M_j \quad \Rightarrow J \cup \{i\} \text{ is in } \Omega. \ \lightning \text{ To maximality of } J. \ \square$$

<u>Theorem 1.2.4</u> For an R-module M TFAE:

(a) M is semi-simple.

(b) M is a sum of simple submodules.

(c) Every submodule of M is a direct summand (i.e. $\forall N \subseteq M$ $\exists N' \subseteq M$ s.t. $M = N \oplus N'$)

<u>Pf</u>: <u>(b) $\Rightarrow$ (a)</u>: Let $(M_i)_{i \in I}$ be a family of simple submods, s.t. $M = \sum_{i \in I} M_i$. By Lemma 1.2.3 (with $N = 0$) there is a $J \subset I$ s.t. $M = 0 \oplus \bigoplus_{j \in J} M_j$, i.e., M is s.s. (def!)

<u>(a) $\Rightarrow$ (c)</u>: Let $(M_i)_{i \in I}$ be a family of simple submods s.t. $M = \bigoplus_{i \in I} M_i$ and let $N \subset M$. Then $M = N + \sum_{i \in I} M_i$. Again, by Lemma 1.2.3. $\exists J \subset I$ s.t. $M = N \oplus \bigoplus_{j \in J} M_j$, i.e., N is a direct summand.

<u>(c) $\Rightarrow$ (b)</u>: <u>Claim</u>: Every <u>submodule</u> $M' \subset M$ also satisfies (c).

Pf of Claim: Let $N' \subset M'$ be a submodule. Then N' is also a submod of M. Thus $\exists N'' \subset M$ with $M = N' \oplus N''$. Then $M' \overset{(!)}{=} N' \oplus (N'' \cap M')$. $\square$

note: $M' = (N' \oplus N'') \cap M'$

Now let $M' \subset M$ be the sum of all simple submodules of M. (with $M' = 0$ if M has no simple submods). By (c), $\exists N \subset M$, s.t.

$$M = M' \oplus N.$$

every proper $N \subsetneq M$ is cont. in max. $N \neq M$

Have to show: $N = 0$.

Assume on contrary, that $N \neq 0$, and $\exists x \neq 0 \in N$. By Rmk 1.2.2.(3(b) applied to $0 \subset Rx$), Rx has a max. submodule N'.

Since $Rx \subset N$ satisfies (c) (by Claim above!), $\exists L \subseteq Rx$ with $Rx = N' \oplus L$. Then $L \cong (Rx)/N'$, and by maximality of N', the module L is simple.

By def: $L \subset M'$, but on the other hand, we have $L \subset Rx \subset N$.

bec. $M' = \sum$ of simples

$\lightning$ to $N \cap M' = 0$. $\square$

Ex: (1) $\mathbb{Z}/4\mathbb{Z}$ is not a semisimple $\mathbb{Z}$-module, because $2\,\mathbb{Z}/4\mathbb{Z}$ is a submodule, which is not a direct summand.

(2) $k \subset k[x]$ is simple module ($k \cong k[x]/(x) \leftarrow$ max. left id)

only simple if $k = \bar{k}$ by weak HNS, if k not alg. closed, simples are extensions of k.

In gen: $k[x]/(p)$, p irred, are simple modules.

Cor. 1.2.5 If M is a semi-simple module, then the same is true for any submodule and every factor module.

Pf: Suppose that M is s.s., and let $N \subset M$ be a submodule. In the proof of Thm 1.2.4 (Claim), we have seen that (c) is inherited by submods, whence N is s.s. and $\exists N' \subset M$ s.t. $M = N \oplus N'$. We have $N' \cong M/N$, so both $N' \cong M/N$ and N are semi-simple. $\square$

Rmk 1.2.6 (1) Every simple module is noetherian and artinian

∴ look at chains of submods: only $\{0\} \subseteq M$ ACC ✓ DCC ✓

(2) Let M be a s.s. module.

(a) M is noetherian $\overset{①}{\Longleftrightarrow}$ M is a finite $\oplus$ of simple modules $\overset{②}{\Longleftrightarrow}$ M is f.g.

(① $\Leftarrow$: ACC ✓, $\Rightarrow$: contradiction
② $\Rightarrow$: a simple mod is f.g: $M \cong R/L$ gen by $\bar{1}$ $\rightsquigarrow$ f. many
$\Leftarrow$: Use Thm 1.2.4 + f.g)

(b) M is indecomposable, i.e., if $M = N_1 \oplus N_2$, then either $N_1 = 0$ or $N_2 = 0$, $\Longleftrightarrow$ M is simple.

<u>Lemma 1.2.7</u> Let $\varphi: M \to M'$ be a nonzero R-module homom.

(1) If M is simple, then φ is injective.

(2) If M' is simple, then φ is surjective.

(3) If M and M' are simple, then φ is an isomorphism.

<u>Pf</u>: Use that $\ker\varphi \subset M$ and $\mathrm{Im}(\varphi) \subset M'$ are submodules. $\square$

e.g. (1): Since M simple : $\ker\varphi = \begin{cases} 0 \Rightarrow \varphi \text{ inj} \\ M \Rightarrow \varphi \equiv 0 \ ↯ \end{cases}$

(2) $\mathrm{Im}(\varphi) = \begin{cases} 0 \Rightarrow \varphi \equiv 0 \ ↯ \\ M' \Rightarrow \varphi \text{ surj} \end{cases}$

<u>Lemma 1.2.8</u> SCHUR'S LEMMA

(1) If M and M' are non-isom. simple R-modules, then $\mathrm{Hom}_R(M, M') = 0$.

(2) If M is simple, then $\mathrm{End}_R(M) = (\{\varphi: M \to M,\ R\text{-lin}\}, +, \circ)$ is a division ring.

<u>Pf</u>: (1): Clear by Lemma 1.2.7. $[0 \neq \varphi \in \mathrm{Hom}_R(M, M') \Longleftrightarrow M \cong M']$

(2) By Lemma 1.2.7, each $\varphi \in \mathrm{End}_R(M)$ is either 0 or an isom.. Then $\exists\, \psi: M \to M$ s.t. $\varphi \circ \psi = \psi \circ \varphi = \mathrm{Id}_M$. Since Id_M is the identity of $\mathrm{End}_R M$, the claim follows. $\square$

<u>Def 1.2.9</u> Let M be an R-module.

(1) Let $\ell(M) = \ell_R(M) := \sup\limits_{\ell \in \mathbb{N} \cup \{\infty\}} \{\exists \text{ chain of submods } M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_\ell = 0\}$

$\hookrightarrow \mathbb{N}$ contains 0!

If $\ell < \infty$, then we say that M has finite length.
(or M is module of f. length)

(2) (a) If $M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_\ell = 0$ $(*)$
with M_{i-1}/M_i simple for each $1 \leq i \leq \ell$, then $(*)$ is called a composition series of length ℓ (for M). ($\Leftrightarrow \forall 1 \leq i \leq \ell$ there is no module M' with $M_{i-1} \supsetneq M' \supsetneq M_i$).

(b) If $M = M_0' \supsetneq M_1' \supsetneq \dots \supsetneq M_k = 0$ $(**)$
is another chain, then $(*)$ and $(**)$ are equivalent if $k = \ell$ and $\exists \sigma \in S_\ell$ s.t. $M_{i-1}/M_i \cong M'_{\sigma(i)-1}/M'_{\sigma(i)}$ $\forall 1 \leq i \leq \ell$. (!)

(c) $(**)$ is a refinement of $(*)$ if $\forall 1 \leq i \leq \ell$, the module $M_i \in \{M_1', \dots, M_k'\}$.

E.g: $k^4 \supsetneq k^2 \supsetneq k \supsetneq 0$
$k^4 \supsetneq k^3 \supsetneq k \supsetneq 0$
equiv by $\sigma = (12)$

Rmk 1.2.10 (1) $\ell(M) = 0 \Leftrightarrow M = 0$ and $\ell(M) = 1 \Leftrightarrow M$ is simple.
(2) If $\ell(M) < \infty$, then M is f. generated.

Pf (2): Let $M = M_0 \supsetneq \dots \supsetneq M_\ell = 0$ comp. series. Pick $x_0 \in M_0 \setminus M_1$, then M_0 is gen. by $\{x_0\} \cup X_1$, where X_1 gen. M_1, by induction $M_0 = \{x_0, \dots, x_{\ell-1}\}$ (since 0 is gen. by $\emptyset$)