Exercises: Let (R,m) be a noeth local ring and $I \subseteq R$ ideal.
Then $\operatorname{embdim}(R) = \operatorname{embdim}(R/I) + \dim_{R}\left(\frac{I+m^2}{m^2}\right)$

Rmk 1.50 Let (R,m) be a noeth. local ring. By Krulls height then $\operatorname{embdim}(R) = \mu(m) \geq \operatorname{height}(m) = \dim(R)$

When m has the smallest possible number of min. generators then R is called regular.

Def 1.51 A noeth. local ring (R,m,k) is a regular lo-

cal ring if m is minimally generated by $\dim(R)$ many elements, that is,

$$\text{embdim}(R) = \dim(R).$$

If $\dim(R) = d$ and $m = (x_1, \dots, x_d)$, the elements $x_1, \dots, x_d$ are called a regular system of parameters.

Ex. 1.52 (a) Any field is a regular local ring: $\dim(k) = 0$ and max ideal is gen by 0 ← no elements. Conversely, any artinian (comm.!) ring is regular $\Leftrightarrow$ it is a field. (Krull-dim)

(b) $k[[x_1, \dots, x_d]]$, k a field is reg. local ring: $\dim = d$ and $x_1, \dots, x_d$ are reg. syst. of parameters

localization of a polyn. ring at hom. max. ideal is also regular: $k[x_1, \dots, x_d]_{(x_1, \dots, x_d)}$

(c) $\mathbb{Z}_{(p)}$, $\mathbb{Z}_{(p)}[x]_{(p,x)}$ are reg. local

(d) The local ring $R = k[[x,y]]/(x^2, xy)$ is not regular: it has dimension 1, but embedding dim 2.

Example 1.53 (!) (R, m) regular. If $x \in m \setminus m^2$, then $R/(x)$ is reg. local ring of $\dim = \dim(R) - 1$.

Thm 1.54 Every regular local ring is a domain.

Pf: left out. [Use induction on $d = \dim(R)$ and prime avoidance].

Reference: Thm 14.3 in [Matsumura]
Prop 2.2.3 in [Bruns-Herzog]

Now we get to a homological characterization of regular local rings: Auslander-Buchsbaum and Serre. These were motivated by the

localization problem: If R is a regular local ring, must R_p be a regular local ring $\forall$ primes p in R?

[Yes - A-B+S!]

This question is answered by the following:

Thm 1.55 [Auslander-Buchsbaum 1957; Serre 1956]: Let (R, m, k) be a noetherian local ring of $\dim(R) = d$. TFAE:

(1) $\operatorname{pdim}(k) < \infty$

(2) Every fin. gen. R-module M has finite proj. dimension.

(3) The maximal ideal m is generated by a regular sequence.

(4) The maximal ideal m is generated by d elements.

(sketch) Full proof see Thm 2.2.7 of [Bruns-Herzog]

Pf: (2) $\Rightarrow$ (1): obvious, as $k \in \operatorname{mod}(R)$. (1) $\Rightarrow$ (2) was shown in Cor 1.18.

(3) $\Rightarrow$ (1): If m is gen. by a regular sequence $\underline{x}$, then the Koszul cx. on $\underline{x}$ is a free resolution of k, by Cor. 1.38. Thus k has pdim equal to $\mu(m)$. By Thm. 1.46 the length $\mu(m)$ of that regular sequence is the height of m, which is d.

(1) $\Rightarrow$ (4): left out (use thm of Fitting-Nakayama on prime avoidance — [Griffo p.31 Thm 1.71] and facts on Tor

Show that $\operatorname{pdim}(k) \leq \dim(R) = d$

(4) $\Rightarrow$ (3): Let $m = (x_1, \dots, x_d)$. If $d = 0$, then $m = (0) = (0)$ and there is nothing to prove (i.e. $R = k$ a field!)

Use induction on $d \geq 1$, and that we have shown that whenever m is generated by $(d-1)$ elements $x_1, \dots, x_{d-1}$, then $x_1, \dots, x_{d-1}$ form a regular sequence.

Since there is at least one min. generator x_1 and $x_1 \in m \setminus m^2$, so by Example 1.53 we know that $R/(x_1)$ is a regular local ring of dimension $d-1$. By induction hypothesis $x_2, \dots, x_d$ form a regular sequence on $R/(x_1)$. Moreover, by Thm 1.54 $R/(x_1)$ is a domain, so x_1 is a regular element in R. We conclude that $x_1, \dots, x_d$ form a regular sequence on R. □

This proof directly shows:

Cor 1.56 Every regular local ring (R, m, k) has $\operatorname{pdim}(k) = \dim(R)$

Therefore, for any f. gen. R-module M, we have
$$\operatorname{pdim}(M) \leq \dim(R).$$
Cor 2.29 in [Bruns-Herzog]

This solves the localization problem:

Cor 1.57 If (R, m, k) is regular local ring, then $R_{\mathfrak{p}}$ is regular for any prime ideal $\mathfrak{p}$ (i.e. $\forall \mathfrak{p} \in \operatorname{Spec}(R)$).

Pf: Set $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $R/\mathfrak{p} \in \operatorname{mod}(R)$, and by Thm 1.54 has $\operatorname{pdim}_R(R/\mathfrak{p}) < \infty$. But localization is an exact functor (i.e. localizing free res. in R gives free res. over $R_{\mathfrak{p}}$) so
$$\operatorname{pdim}_{R_{\mathfrak{p}}}((R/\mathfrak{p})_{\mathfrak{p}}) = \operatorname{pdim}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}) < \infty.$$
This is shown e.g. in [Illetsumeru] Thm I.4.2 p.23
But $\mathfrak{p}_{\mathfrak{p}}$ is the max. ideal in the local ring $R_{\mathfrak{p}}$ so $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ is the residue field $k_{\mathfrak{p}}$. Again, Thm 1.54 (3)$\Rightarrow$(4) tells us that $R_{\mathfrak{p}}$ is a regular local ring. □

This allows us to make a global definition of reg. rings:

Def 1.58 A noetherian ring R is regular if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Rmk 1.59: To show that a particular ring is regular, it is sufficient to show that R_m is a reg. local ring for every max. ideal m of R: given any $\mathfrak{p} \in \operatorname{Spec}(R)$, there exists a max. ideal $m \supseteq \mathfrak{p}$, and $R_{\mathfrak{p}} = (R_m)_{\mathfrak{p}}$ is a localization of the local ring R_m, which by Cor. 1.57 must be a reg. local ring.

Hilbert's syz. theorem (Thm 1.39) says that any graded fin. gen. module over the polynomial ring $k[x_1, \dots, x_n]$ is of fin. pdim, but this is not quite enough to conclude that $k[x_1, \dots, x_n]$ is regular.
One case is easy (with what we have shown:)

Prop 1.60: $k[x_1,\dots,x_d]$ is regular if k is algebraically closed.

Pf: By Hilbert's Nullstellensatz, the max. ideals in $k[x_1,\dots,x_d]$ are of the form $(x_1-a_1,\dots,x_d-a_d)=m_a$, which are all generated by d elements. And they all have height d. $\Rightarrow R_{m_a}$ is regular local ring $\forall m$ maximal, and R is a regular ring. □

The general case is more complicated, we state the most general result (without proof) – see [Bruns-Herzog] Thm 2.2.13

Thm 1.61 A noeth. ring R is regular $\Leftrightarrow$ the polynomial ring $R[x_1,\dots,x_d]$ is regular.

Def 1.62 Let R be a noeth. ring. The global dimension of R is $\operatorname{gldim}(R) := \sup\{\operatorname{pdim}_R(M) : M \text{ is a f. gen. } R\text{-module}\}$.

Rmk: Can indeed take sup over all $M \in \operatorname{Mod}(R)$

Now a restatement of Auslander-Buchsbaum-Serre:

Thm 1.63 Let (R, m, k) be a noetherian local ring. Then
$\operatorname{pdim}(k) < \infty \Leftrightarrow \operatorname{gldim}(R) < \infty$.
Moreover, if $\operatorname{gldim}(R) < \infty$, then we must have $\operatorname{gldim}(R) = \dim(R)$

More generally: $\operatorname{gldim}(R) < \infty$ characterizes regular rings!

Note: This is much more complicated for noncommutative noetherian rings!

Lemma 1.64 If R is a regular ring, then $\operatorname{gldim}(R) \le \dim(R)$.

Pf: If $\dim(R) = \infty$, then there is nothing to prove.
So assume that $\dim(R) = d < \infty$. Let $F_\bullet$ be a free resolution for $M \in \operatorname{mod}(R)$, let $\mathfrak{p} \in \operatorname{Spec}(R)$, and let

$$\Omega := \ker(F_{d-1} \xrightarrow{\partial_{d-1}} F_{d-2}).$$

Localization is exact, so $F_\bullet \otimes_R R_\mathfrak{p}$ is a free resolution of $M_\mathfrak{p}$ and $\Omega_\mathfrak{p}$ is the kernel of $\partial_{d-1} \otimes_R R_\mathfrak{p}$. By assumption, $R_\mathfrak{p}$ is regular local ring with $\dim(R_\mathfrak{p}) \leq d$, so by Thm 1.54 $\operatorname{pdim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq d$. Any resolution for $M_\mathfrak{p}$ is a direct sum of a min. resolution and a trivial complex, and must be of length $\leq d$. Thus $\Omega_\mathfrak{p}$ must be a free module over $R_\mathfrak{p}$.

This holds $\forall \mathfrak{p} \in \operatorname{Spec}(R)$, so Ω is locally free. By a thm in hom. algebra, Ω must be projective, and thus the complex

$$0 \to \Omega \to F_{d-1} \to \dots \to F_0$$

(see Stacks project!)

is a proj. res. for M of length $d \Rightarrow \operatorname{pdim}_R(M) \leq d$. $\square$

Let $d \leq \infty$.

Thm 1.65 A noetherian ring R is regular of $\dim(R) = d \iff \operatorname{gldim}(R) = d$. If $\dim(R) = \infty$, then $\Rightarrow$ still holds.

Pf $\Rightarrow$: If R is regular and $\dim(R) = d < \infty$, then Lemma 1.64 says that $\operatorname{gldim}(R) \leq d$. Let $\mathfrak{m}$ be a max. ideal of height d. By assumption $R_\mathfrak{m}$ is a reg. local ring with $\dim(R_\mathfrak{m}) = d$. Since localization is exact, localizing a finite min. free res. for $R/\mathfrak{m}$ gives us a finite res. of $(R/\mathfrak{m})_\mathfrak{m} = R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$ which may not be minimal.

$$\Rightarrow d \geq \operatorname{gldim}(R) \geq \operatorname{pdim}_R(R/\mathfrak{m}) \geq \operatorname{pdim}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}) = \dim(R_\mathfrak{m}) = d$$

$$\Rightarrow \dim(R) = d.$$

If $\dim(R) = \infty$, assume for a contradiction that $\operatorname{gldim}(R) = n < \infty$. As before, localizing in a $\mathfrak{p} \in \operatorname{Spec}(R)$ yields $R_\mathfrak{p}$ is a regular local ring with $\dim(R_\mathfrak{p}) \leq n$. By Hilbert-B $\operatorname{gldim}(R_\mathfrak{p}) = \dim(R_\mathfrak{p}) \leq n$.

This holds for any $\mathfrak{p} \in \operatorname{Spec}(R) \Rightarrow \dim(R) = \sup_{\mathfrak{p} \in \operatorname{Spec}(R)} \{\text{height}(\mathfrak{p})\} \leq n$. $\lightning$ to $\dim(R) = \infty$.

$\Leftarrow$: Assume that R has $\operatorname{gldim}(R) < d < \infty$. Then $\forall$ max. ideals $\mathfrak{m}$,

$\mathrm{pdim}_R(R/\mathfrak{m}) \leq d$. Since localization is exact, and fin. gen. projectives over a noeth. ring are locally free, localizing a finite projective res. for $R/\mathfrak{m}$ gives us a finite proj. res. for $(R/\mathfrak{m})_{\mathfrak{m}} \cong R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$, and thus $R_{\mathfrak{m}}$ is regular local by Auslander-Buchsbaum-Serre. Since $\mathfrak{m}$ was any max. ideal, by def. R is a regular ring.

Our argument also shows that (32)

$$d = \mathrm{gldim}(R) \geq \mathrm{pdim}_R(R/\mathfrak{m}) \geq \mathrm{pdim}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}) = \dim(R_{\mathfrak{m}}) = \mathrm{height}(\mathfrak{m}).$$

In part., $\dim(R)$ is finite, bounded above by d. By Lemma 1.64 $d = \mathrm{gldim}(R) \leq \dim(R)$

$\Rightarrow \dim(R) = \mathrm{gldim}(R)$. □