Rmk 1.36 Suppose that $r \in R$ is s.t. $rM \neq M$ for some $M \in \mathrm{mod}(R)$.
Claim: r is regular on $M \iff H_1(r;M) = 0$, i.e. 1st Koszul hom vanishes. $\mathrm{kos}(r;M)$: $0 \to M \xrightarrow{\cdot r} M \to 0$, $1 \mapsto r$, $1 \mapsto r \cdot 1 = r$

Indeed: $H_1(r;M) = \ker(M \xrightarrow{\cdot r} M) = (0 :_M r)$.
$H_0(r;M) = M/rM$
By def: r is regular on $M \iff (0 :_M r) = 0$.

In fact, the Koszul cx. on a regular sequence is exact in all positive degrees:

Thm 1.37 If $\underline{x} = x_1, \dots, x_n \in R$ is a regular sequence on the R-module M, then $H_i(\underline{x};M) = 0 \ \forall i > 0$.

Proof: We use induction on length of reg. sequence. Note that $n=1$ is Rmk 1.36.
Let $n \geq 1$, and suppose that all regular sequences on M of length n have the property that $H_i(y;M) = 0 \ \forall i > 0$.
Let $\underline{x} = x_1, \dots, x_{n+1}$ be a regular sequence of M. Then $x_1, \dots, x_n$ is also a regular seq. on M, so by induction hypothesis

$$H_i(x_1, \dots, x_n; M) = 0 \quad \forall i > 0.$$

In Rmk 1.29 we viewed $\mathrm{kos}(\underline{x};M)$ as the cone of the multiplication by x_{n+1} on $\mathrm{kos}(x_1, \dots, x_n; M)$, and got long exact

sequence in homology:

$$H_i(x_{11\cdots1}x_n;M) \xrightarrow{\varphi} H_i(x_{1,\cdots,}x_{n+1};M) \xrightarrow{\psi} H_{i-1}(x_{11\cdots1}x_n;M).$$

for all $i \geq 2$ the two H_i on left and right vanish, (by ind. hypothesis), and thus $H_i(x_{11\cdots1}x_{n+1};M)=0$ as well.
We conclude that $H_i(x_{11\cdots1}x_{n+1};M)=0 \ \forall i \geq 2$.

Also in Rmd 1.29: have s.e.s:

$$(*)\quad 0 \to H_1(x_{11\cdots1}x_i;M)/x_{i+1}H_1(x_{1\cdots1}x_i;M) \to H_1(x_{1\cdots1}x_{i+1};M) \to \operatorname{ann}_{H_0(x_{11\cdots1}x_i;M)}^{(x_{i+1})} \to 0.$$

By induction hypothesis: $H_1(x_{11\cdots1}x_n;M)=0$. Since x_{n+1} is regular on $M/(x_{11\cdots1}x_n)M = H_0(x_{11\cdots1}x_n;M)$,
we must have $\operatorname{ann}_{H_0(x_{1\cdots1}x_n;M)}(x_{n+1})=0$.
$\Rightarrow$ in $(*)$ we have s.e.s. $\overset{(*)}{0\to 0} \to H_1(x_{11\cdots1}x_{n+1};M) \to 0 \to 0$
$\Rightarrow H_1(x_{11\cdots1}x_{n+1};M)=0$. □

Cor 1.38 If $x_{11\cdots1}x_n$ is a regular sequence on R, then the Koszul complex on $x_{11\cdots}x_n$ is a free resolution for $R/(x_{11\cdots1}x_n)$. Moreover, if (R,m,k) is local/graded, then $\mathrm{Kos}(x_{11\cdots1}x_n)$ is a minimal free resolution for $R/(x_{11\cdots1}x_n)$.

Pf: By Thm 1.37, $P_0 = \mathrm{Kos}(x_{11\cdots1}x_n)$ has $H_i(P_0)=0 \ \forall i>0$.
This is a ex. of free modules, and thus a free resolution of $H_0(P)$, which by Thm 1.28 (2) is $R/(x_{11-}x_n)$.
In the local/graded case, the assumption that $\underline{x}$ is a regular sequence guarantees that $x_i \in m \ \forall i$. Since all nonzero entries in the differentials (under the std basis for the Koszul ex.) are of the form $\pm x_i$, we conclude that the resolution is minimal. □

Now we are ready to prove the third big thm of Hilbert in comm. algebra (1st: Hilbert basis thm, 2nd: Hilbert Nullstellensatz)

Thm 1.39 (Hilbert's syzygy theorem) Every finitely generated graded module M over the polyn. ring $R = k[x_1, \dots, x_m]$ over a field k has finite projective dimension. In fact,

$$\operatorname{pdim}(M) \leq n.$$

Proof: By Cor. 1.38, the Koszul c. on the regular sequence $x_1, \dots, x_n$ is a minimal free resolution of $k = R/(x_1, \dots, x_n)$, so $\operatorname{pdim}(k) = n$. By Cor 1.18 every fin. generated R-module has $\operatorname{pdim}(M) \leq \operatorname{pdim}(k) = n$. $\square$

Now we come to the converse of Thm 1.37: we will see that the vanishing of H_i indeed characterizes regular sequences (over nice enough rings):

Thm 1.40 Let (R, m, k) be local/graded. Let $M \neq 0$ be a fin. generated R-module, and $\underline{x} = x_1, \dots, x_n \in m$. In the graded case, we assume that M is graded and $x_1, \dots, x_n$ are all homogeneous. TFAE:

(1) $H_i(\underline{x}; M) = 0 \quad \forall i > 0$

(2) $H_1(\underline{x}; M) = 0$

(3) $\underline{x}$ is a regular sequence on M.

Pf: (1) $\Rightarrow$ (2) is obvious and (3) $\Rightarrow$ (1) is Thm 1.37. We show (2) $\Rightarrow$ (3) here, by induction on the length of $\underline{x}$. The case $n = 1$ is Rmk 1.36. So suppose the statement holds for some $n \geq 1$.

Recall from Rmk 1.29 the s.e.s.

$$0 \to \frac{H_1(x_1, \dots, x_n; M)}{x_{n+1} \cdot H_1(x_1, \dots, x_n; M)} \to H_1(x_1, \dots, x_{n+1}; M) \to \operatorname{ann}_{H_0(x_1, \dots, x_n; M)}(x_{n+1}) \to 0.$$

By our assumption: $H_1(x_1, \dots, x_n; M) = 0$, so all terms in the s.e.s above must vanish.

The left term gives us: $x_{n+1} H_1(x_1,\dots,x_n;M) = H_1(x_1,\dots,x_n;M)$.
But $H_1(x_1,\dots,x_n;M)$ is a f. gen. R-module by Thm 1.28. and $x_{n+1} \in \mathfrak{m}$, so $H_1(x_1,\dots,x_n;M) = 0$ by Nakayamas lemma. (also graded version)
By induction hypothesis, $x_1,\dots,x_n$ is a reg. sequence on M. Note that then Nak. also guarantees that $M/(\underline{x}) M \neq 0$, $\underline{x} = x_1,\dots,x_{n+1}$. (1st cond. of reg seq.)

Finally: the vanishing of right term: $\operatorname{ann}_R H_0(x_1,\dots,x_n;M)(x_{n+1}) = 0$
By def, x_{n+1} is regular on $H_0(x_1,\dots,x_n;M) = M/(x_1,\dots,x_n)M$.
Thus $x_1,\dots,x_{n+1}$ is a reg. sequ. on M. □

Now is a cor., we see that over local/graded rings, the order of elements in a regular sequence does not matter.

Cor 1.41 Let $(R,\mathfrak{m},k)$ be local/graded. Let M be a f. gen. R-module and consider $\underline{x} = x_1,\dots,x_n \in \mathfrak{m}$. In the graded case, assume that M is graded and all x_i are homogeneous. If the sequence $\underline{x}$ is regular on M, so are all its permutations.

(sketch)
Pf: If $x_1,\dots,x_n$ is a reg. seq., by Thm 1.37 gives us

$$H_i(x_1,\dots,x_n;M) = 0 \quad \forall i > 0.$$

One can show(!) that Koszul cx. on $\underline{x}$ agrees with Koszul cx. on any permutation of $\underline{x}$, so all of the corr. homology groups must be $=0$. By Thm. 1.40. any permutation of $\underline{x}$ is a regular sequence. □

One can extend this to any ring and any module under reasonable assumptions:

Lemma 1.42 Let R be a ring and M an R-module. If x, y is a regular seq. on M and y is reg. on M, then y, x is a reg. seq. on M.

Pf: Left out. (Elementary!)

Lemma 1.43 ("Geometric nature of depth" – Eisenbud Cor 17.8) Let (R,m) be a noeth. local ring. If $x_1,\dots,x_n$ is a reg. sequence on a f.gen. module M, then $x_1^{a_1},\dots,x_n^{a_n}$ is as well for any integers $a_i \geq 1$, $i=1,\dots,n$.

Pf: Let $n=1$. Suppose $x=x_1$ is a regular element on M. Given an $m \neq 0 \in M$ and $a \geq 1$. Then: $x^a m = 0 \to x \cdot x^{a-1} m = 0$.
Since x is regular on M, we must have $x^{a-1}m = 0$. Repeating this $(a-1)$ times, we conclude that $xm=0$, and $m=0$.
Now consider any $n \geq 1$. Since x_n is a regular element on $M/(x_1,\dots,x_{n-1})M$ by the length 1 case, we can now say $x_n^{a_n}$ is regular on $M/(x_1,\dots,x_{n-1})$. Therefore, $x_1,\dots,x_{n-1},x_n^{a_n}$ is regular on M. By Cor. 1.41, we are allowed to permute the elements in our sequence. Now switch the order and repeat the argument with each x_i, until we conclude that $x_1^{a_1},\dots,x_n^{a_n}$ is also regular on M. □

Tweak out:
Now we can give examples where Koszul homology is neither left or right exact:

Example 1.44 Let $R=k[x]$ and consider $M=R/(x^2)$. Note that x is regular on R but not on M, so

$$H_1(x;R)=0 \quad \text{and} \quad H_1(x;M)\neq 0.$$

On the other hand $R \twoheadrightarrow M$ is surjective, while the induced map on the first Koszul homology

$$0=H_1(x;R) \to H_1(x;M) \neq 0$$

cannot be surjective. Thus $H_1(x;-)$ is not right exact.

Example 1.45 Let $R=k[x]$. Then x is regular on R, so we get a short exact sequence $0 \to R \xrightarrow{x} R \to R/(x) \to 0$
On the one hand, $H_0(x;R)=R/(x)$, but

$$H_0(x; R \xrightarrow{x} R) = R/(x) \to R/(x),$$

is the 0-map. In particular, $H_0(x;-)$ is not left exact.

More generally, $H_0(\underline{x};-)$ is naturally isomorphic to $-\otimes_R R/(\underline{x})$, which is typically not left exact (but in fact right exact!).

recall: $ht(I) = \min\{ht(\mathfrak{p}) : \mathfrak{p} \text{ is prime in } R \text{ and } I \subseteq \mathfrak{p}\}$
$ht(\mathfrak{p})$ = length of longest chain of p.i.'s $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$.

Thm 1.46 If $x_1, \dots, x_n$ is regular sequence on R, then $\text{height}(x_1, \dots, x_n) = n$. Moreover, any min. prime of $(x_1, \dots, x_n)$ has height $= n$.
(min p.i. containing $(x_1, \dots, x_n)$)

Pf: By induction on n. When $n=1$, then $x = x_1$ is regular $\iff x_1$ is not a zero divisor on R. This means that x_1 is not in any of the associated primes of R (← see Commely.) and in part. not in any of the minimal primes of R. (← those have ht = 0.)
Therefore, any prime containing x must have height at least 1, so $ht(x_1) \geq 1$. By Krull's height Thm, we have $ht(x_1) \leq 1$, thus $ht(x_1) = 1$.
Now ind. step: assume that for an $n \geq 1$ all reg. seq. of length n have $ht(x_1, \dots, x_n) = n$. If $x_1, \dots, x_{n+1}$ is a reg. sequ. on R, then x_{n+1} is regular on the quotient $R/(x_1, \dots, x_n)$. By the base case, the ideal $(x_1, \dots, x_n, x_{n+1})/(x_1, \dots, x_n)$ in this ring has $ht = 1$.
By ind. hyp. $ht(x_1, \dots, x_n) = n$. $\Rightarrow ht(x_1, \dots, x_{n+1}) = n+1$.

We have shown $h(x_1, \dots, x_n) = n \Rightarrow$ every min. prime of $(x_1, \dots, x_n)$ has ht at least n. But Krull's height theorem also says that every min. prime of $(x_1, \dots, x_n)$ has ht at most n. $\Rightarrow$ Every min. prime has height exactly n. □

The converse does not hold in general:

Ex 1.47 Let $R = k[x,y]/(x^2, xy)$. The element y is not in the unique minimal prime $(x)/(x^2, xy)$ of R, so $ht(y) = 1$.

However y is not regular on R.

Recall: In CM rings: $\mathrm{ht}(x_1,\dots,x_n) = n \iff x_1,\dots,x_n$ form a reg. sequence

Rmk 1.48 In local/graded setting, Cor. of Thm 1.46 is that if $\underline{x}$ is a reg. seq., then $\underline{x}$ is a min. gen. set for $(\underline{x})$. (!)

1.5. Regular rings: "next best rings to fields"

Def 1.49 The embedding dimension of a noetherian local ring $(R, \mathfrak{m}, k)$ is

$$\operatorname{embdim}(R) := \mu(\mathfrak{m}) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$$

This is coming from geometry: If $Q = k[x_1,\dots,x_d]$, k field, and I homogeneous ideal in Q. Geometry: $V(I) \subseteq \operatorname{Spec}(Q)$ "$\mathbb{A}^d_k$", $\operatorname{Spec}(Q/I)$

dimension of $V(I)$ is Krull-dim of R/I.
If $I \subseteq (x_1,\dots,x_d)^2$, then one can show that we cannot embed $V(I)$ in $\mathbb{A}^n_k$ for any $n < d$. Thus d is the embedding dim of $V(I)$. Note now that d is the min number of generators of the unique homogeneous maximal ideal $(x_1,\dots,x_d)/I$ of R.