Recall:

Thm 1.28 Let R be a ring and $I=(x_1,\dots,x_n)$ be an ideal, and M be an R-module.

(1) $H_i(\underline{x};M)=0$ whenever $i<0$ or $i>n$.

(2) $H_0(\underline{x};M)=M/IM$.

(3) $H_n(\underline{x};M)=(0:_M I)=\operatorname{ann}_M(I)=\{m\in M: m\cdot x=0 \ \forall x\in I\}$

"$\{m\in M: x_1 m=\dots=x_n m=0\}$" $=\operatorname{Hom}_R(R/(x_1,\dots,x_n),M)$

(4) Every Koszul homology module $H_i(\underline{x};M)$ is killed by $\operatorname{ann}_R(M):=\{r\in R: rm=0 \ \forall m\in M\}$

(5) Every Koszul homology module $H_i(\underline{x};M)$ is killed by I.

(6) If M is a Noetherian R-module, so is $H_i(\underline{x};M)$ $\forall i$.

(7) For every i, $H_i(\underline{x};-)$ is a covariant additive functor $R\text{-Mod}\to R\text{-Mod}$.

(8) Every s.e.s. of R-module

$$0\to A\to B\to C\to 0$$

gives rise to a long exact sequence of homology,

$$\dots\to H_1(\underline{x};C)\to H_0(\underline{x};A)\to H_0(\underline{x};B)\to H_0(\underline{x};C)\to 0.$$

↑ VO 14/10/21

Proof (1) Immediate from the definition, because Koszul c. is only nonzero in homological degrees 0 to n.

(2) Rmk 1.26 tells us that

$$H_0(\underline{x}; M) = \operatorname{coker}\left(M^{n} \xrightarrow{(x_1 \cdots x_n)} M\right) = M/IM.$$

(3) Rmk. 1.26 tells us again that

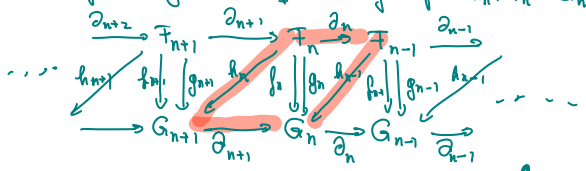
$$H_n(\underline{x}; M) = \ker\left(M \xrightarrow{\binom{x_1}{\vdots}} M^n\right)$$
$$= \{ m \in M : mx_1 = \dots = mx_n = 0 \}$$
$$= (0 :_\mu I).$$

(4) In each hom. degree the Koszul complex is a direct sum of copies of M. So the modules in the ass. are themselves already killed by $\operatorname{ann}_R(M)$ (and as well $\ker(M \to M^{\oplus \beta}/\operatorname{im}(\beta))$

(5) We are proving a stronger statement: show that mult. by $a \in I$ on $\operatorname{kos}(\underline{x}; M)$ is nullhomotopic, which proves that it is the zero map on $H_i(\underline{x}; M)$.

From hom. alg.:

Def 2.3.8 Let $f, g : F_\bullet \to G_\bullet$ be maps of complexes. A chain homotopy between f and g is a sequence of maps $h_n : F_n \to G_{n+1}$



s.t. $h_{n-1} \circ \partial_n + \partial_{n+1} \circ h_n = f_n - g_n \quad \forall n$

f nullhom: $f \simeq 0$.

Lemma 2.3.9 Let $f, g : (F_\bullet, \partial_\bullet^F) \to (G_\bullet, \partial_\bullet^G)$ be maps of cxs. If f is homotopic to g, then $H_n(f) = H_n(g)$ $\forall n$.

In particular, every nullhomotopic map induces the 0-map on homology.

Enough to show that mult. by a is nullhomotopic on $\operatorname{kos}(\underline{x})$ since additive functors preserve homotopy relations.

To do this, use mult. structure of the Koszul complex given by our description of the Koszul cx. by the exterior powers.

Let $a \in I = (x_1, \dots, x_n)$. Then $a = a_1 x_1 + \dots + a_n x_n$.

Consider the chain map: $s_a : \operatorname{kos}(\underline{x}) \to \Sigma^{-1}(\operatorname{kos}(\underline{x}))$ (hom. shift (shift on der. cat.))

$$s_a(e_{i_1} \wedge \dots \wedge e_{i_t}) \mapsto \sum_{j=1}^{n} a_j \cdot e_j \wedge e_{i_1} \wedge \dots \wedge e_{i_t}$$

$$\begin{array}{ccc} \bigwedge^k R^n & \to & \bigwedge^{k-1} R^n \\ \downarrow & & \downarrow s_a \\ \bigwedge^k R & \to & \bigwedge^{k-1} R^n \end{array}$$

shift [-1] to the right (Σ^{-1})

Claim: s_a is a nullhomotopy for the map of cx. $kos(\underline{x}) \to kos(\underline{x})$ given by mult. by a in every component:

$$\begin{array}{ccccccc} \cdots & \to & \Lambda^{k+1}R^n & \xrightarrow{\partial_k} & \Lambda^k R^n & \xrightarrow{\partial_{k-1}} & \Lambda^{k-1}R^n \to \cdots \\ & & \downarrow \cdot a & \swarrow s_a & \downarrow \cdot a & \swarrow s_a & \downarrow \cdot a \\ \cdots & \to & \Lambda^k R^n & \xrightarrow{\partial_k} & \Lambda^{k-1}R^n & \to & \Lambda^{k-2}R^n \to \cdots \end{array}$$

i.e. show $s_a \partial_{k-1} + \partial_k s_a = \cdot a$

Enough to show it on basis elements, i.e. check that

$$s_a \partial(e_{i_1} \wedge \cdots \wedge e_{i_k}) + \partial s_a(e_{i_1} \wedge \cdots \wedge e_{i_k}) = a e_{i_1} \wedge \cdots \wedge e_{i_k}$$

Indeed, we have $s_a \partial(e_{i_1} \wedge \cdots \wedge e_{i_k}) = s_a\left(\sum_{j=1}^{k} (-1)^{j+1} x_{i_j} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}\right)$

$$= \sum_{j=1}^{n} \sum_{l=1}^{k} (-1)^{l+1} a_j x_{i_l} e_j \wedge e_{i_1} \wedge \cdots \wedge \hat{e}_{i_l} \wedge \cdots \wedge e_{i_k})$$

and $\partial s_a(e_{i_1} \wedge e_{i_k}) = \partial\left(\sum_{j=1}^{n} a_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}\right)$

$$= \sum_{j=1}^{n} \sum_{l=1}^{k} (-1)^{l+2} a_j x_{i_l} e_j \wedge e_{i_1} \wedge \cdots \wedge \hat{e}_{i_l} \wedge \cdots \wedge e_{i_k}$$

$$+ \sum_{j=1}^{n} a_j x_j e_{i_1} \wedge \cdots \wedge e_{i_k}$$

$$= -s_a \partial(e_{i_1} \wedge \cdots \wedge e_{i_k}) + \sum_{j=1}^{n} a_j x_j e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Since $a_1 x_1 + \cdots + a_n x_n = a \Rightarrow (s_a \partial + \partial s_a)(e_{i_1} \wedge \cdots \wedge e_{i_k}) = a e_{i_1} \wedge \cdots \wedge e_{i_k}$.

(6) If M is noetherian, so is M^k for all k, as well as any submodules and quotients of $M^k \Rightarrow$ since each $H_i(\underline{x}; M)$ is a quotient of a direct sum of copies of M, it must be noetherian.

(7) Given a R-module homom. $f: M \to N$, get an induced map $kos(f): kos(\underline{x}; M) \to kos(\underline{x}; N)$

by taking $kos(\underline{x}) \otimes f$, so we set $H_i(\underline{x}; f) := H_i(kos(\underline{x}) \otimes f)$

It is immediate that this assignment takes the identity on M to itself and (exercise!) this diff. preserves homom=

orphisms.

(8) Given any cx. of free R-modules $F_\cdot$, tensoring with F is an exact functor from: R-Mod $\to$ Ch(R).
In part, this applies to $F_\cdot = Kos(\underline{x})$, we get s.e.s. of complexes
$0 \to Kos(\underline{x}) \otimes_R A \to Kos(\underline{x}) \otimes_R B \to Kos(\underline{x}) \otimes_R C \to 0$
$\rightsquigarrow$ get long exact sequence of homology ($\to$ see Hom. algebra, e.g. [Matsumura, Appendix B "complexes"]) □

In general, Koszul homology functor is neither left nor right exact

Not in detail

Rmk 1.29 From the iterative description of the Koszul cx., i.e.,
$$Kos(x_{1,\dots,n+1}; M) = Kos(x_{1,\dots,n}; M) \otimes Kos(x_{n+1}).$$
set $C := Kos(x_{1,\dots,n}; M)$ and by Def. of tensor product of cx. we have $[Kos(x_{1,\dots,n+1}; M)]_i = C_{i-1} \otimes_R R \oplus C_i \otimes_R R \cong C_{i-1} \oplus C_i$.
One can write down the differential on $Kos(x_{1,\dots,n+1}; M)$ in terms of ∂ of C: given $a \in C_{i-1}$, $b \in C_i$, $r \in R$ (in hom. deg 1) and $s \in R$ (in hom deg 0), then:
$$\partial(a \otimes r + b \otimes s) = \partial(a) \otimes r + (-1)^{i-1} a \otimes (x_{n+1} r) + \partial(b) \otimes s + \underbrace{(-1)^i b \otimes 0}_{=0}$$

$$\partial_i \begin{pmatrix} \partial_C & 0 \\ (-1)^{i-1} x_{n+1} & \partial_C \end{pmatrix} : \begin{matrix} C_{i-1} \\ \oplus \\ C_i \end{matrix} \xrightarrow[(-1)^{i-1} x_{n+1}]{\partial_C} \begin{matrix} C_{i-2} \\ \oplus \\ C_{i-1} \end{matrix}.$$

This is the cone of the map $Kos(x_{1,\dots,n-1}; M) \xrightarrow{x_{n+1}} Kos(x_{1,\dots,n-1}; M)$ given by multiplication by x_{n+1} in every degree. This cone comes together with a s.e.s.

hom. degree shift

$$0 \to Kos(x_{1,\dots,n}; M) \to Kos(x_{1,\dots,n+1}; M) \to \Sigma^{-1} Kos(x_{1,\dots,n-1}; M) \to 0$$

From this s.e.s. get a long exact sequence of homology (!) (Here connecting homomorphism is the map in homology induced by multiplication by x_{n+1}). The deg n piece of

That long exact sequence is:

$$\to H_i(x_1,\dots,x_{n+1};M) \to H_{i-1}(x_1,\dots,x_n;M) \xrightarrow{\cdot x_{n+1}} H_{i-1}(x_1,\dots,x_n;M)\to\dots$$

(i-th deg of this)

Let's look at the sequence:

$$H_i(x_1,\dots,x_n;M) \xrightarrow{\cdot x_{n+1}} H_i(x_1,\dots,x_n;M) \xrightarrow{\varphi} H_i(x_1,\dots,x_{n+1};M) \xrightarrow{\psi} H_{i-1}(x_1,\dots,x_n;M)$$

By exactness $\ker(H_i(x_1,\dots,x_n;M)\xrightarrow{\varphi} H_i(x_1,\dots,x_{n+1};M))$

$= \operatorname{im}(H_i(x_1,\dots,x_n;M)\xrightarrow{\cdot x_{n+1}} H_i(x_1,\dots,x_n;M)) = x_{n+1}\cdot H_i(x_1,\dots,x_n;M).$

By 1st isom. thm φ induces an injection $\bar\varphi$:

$$H_i(x_1,\dots,x_n;M)/x_{n+1}H_i(x_1,\dots,x_n;M) \xrightarrow{\bar\varphi} H_i(x_1,\dots,x_{n+1};M)$$

with $\operatorname{im}(\varphi)=\operatorname{im}(\bar\varphi)$. Now use exactness in 1st isom thm + exactness:

$$\operatorname{coker}(\bar\varphi) \overset{\text{def}}{=} \frac{H_i(x_1,\dots,x_{n+1};M)}{\operatorname{im}\varphi} \overset{\text{exactness}}{=} H_i(x_1,\dots,x_{n+1};M)/\ker\psi \overset{\text{1st isom}}{\cong} \operatorname{im}\psi$$

and $\operatorname{im}(\psi) \overset{\text{def}}{=} \operatorname{im}(H_i(x_1,\dots,x_{n+1};M)\xrightarrow{\psi} H_{i-1}(x_1,\dots,x_n;M))$

$\overset{\text{exact}}{=} \ker(H_{i-1}(x_1,\dots,x_n;M)\xrightarrow{\cdot x_{n+1}} H_{i-1}(x_1,\dots,x_n;M))$

$= \operatorname{ann}_{H_{i-1}(x_1,\dots,x_n;M)}(x_{n+1}).$

Get s.e.s. induced by $\bar\varphi$. (This will be used later.)

$$0 \to \frac{H_i(x_1,\dots,x_n;M)}{x_{n+1}H_i(x_1,\dots,x_n;M)} \to H_i(x_1,\dots,x_{n+1};M) \to \operatorname{ann}_{H_{i-1}(x_1,\dots,x_n;M)}(x_{n+1}) \to 0$$

Def 1.30 Let M be an R-module and consider a sequence of nonunit elements $f=(f_1,\dots,f_n)\in R$. The *i-th Koszul cohomology on f with coefficients in M*, also called *Koszul cohomology on M w.r.t. f*, is the R-module

$$\operatorname{kos}^i(f;M) := H^i(\operatorname{Hom}_R(\operatorname{kos}(f),M)).$$

Write $\operatorname{kos}^i(f) := H^i(\operatorname{Hom}_R(\operatorname{kos}(f);R))$.

This does not add any more information however: the Koszul complex is *self dual*:

cf. Prop 1.6.10 in [Bruns-Herzog]

Thm 1.31 Let R be any ring and let $x=x_1,\dots,x_n\in R$. The Koszul c. $\operatorname{kos}(x)$ is isomorphic to its dual $\operatorname{Hom}_R(\operatorname{kos}(x),R)$.

More generally: $Kos(\underline{x};M) \cong Hom_R(Kos(\underline{x}),M)$.

In particular: $H_i(\underline{x};M) \cong H^{n-i}(Hom_R(Kos(\underline{x}),M))$

Write as $H_i(\underline{x};M)$ as in Rmk — $H^{n-i}(\underline{x};M)$

LEAVE OUT

I Proof: (Sketch)

In general for any two modules $M,N \in Mod(R)$ the not. isom: $R \otimes_R M \cong M$ leads to a Hom-iom

$$Hom_R(N,R) \otimes_R M \xrightarrow{\psi} Hom_R(N,M) \quad (!)$$
$$f \otimes m \mapsto (n \mapsto f(n) \otimes m)$$

When N is f.g. free R-module, then ψ is an isom. (Exercise!)

So $\forall d \geq 1$: $Hom_R(R^d,R) \otimes_R M \cong Hom_R(R^d,M)$.

This isom. is natural, thus induces an isom.

$$Hom_R(Kos(\underline{x},R) \otimes M \cong Hom_R(Kos(\underline{x}),M)$$

Thus it is sufficient to construct an isom

$$Kos(\underline{x}) \xrightarrow{\omega} Hom_R(Kos(\underline{x}),R).$$

note here: $(\wedge^k R^n)^* = \wedge^{n-k}(R^n)^*$ and $(R^n)^* \cong R^n$

This map ext. to a comm. diagram:

$$\begin{array}{ccccccccccc} Kos(\underline{x}) & : & 0 & \to & \wedge^n R^n & \xrightarrow{\partial} & \wedge^{n-1} R^n & \xrightarrow{\partial} & \cdots & \to \wedge^1 R^n \to R \to 0 \\ \downarrow & & & & \downarrow \omega_n & & \downarrow \omega_{n-1} & & & \downarrow \omega_1 \quad \downarrow \omega_0 \\ Hom_R(Kos(\underline{x}),R) & : & 0 & \to & R & \xrightarrow{\partial^*} & \wedge^1 R^n & \xrightarrow{\partial^*} & \cdots & \to \wedge^{n-1} R^n \to \wedge^n R \to 0 \end{array}$$

For each $I = \{i_1,\dots,i_d\} \subseteq \{1,\dots,n\}$ write e_I for hom. deg. d basis element of $Kos(\underline{x})$: $e_I = e_{i_1} \wedge \cdots \wedge e_{i_d} \in \wedge^d R^n$.

Define $\omega_d(e_I) \in Hom_R(\wedge^d R^n, (\wedge^n R^n)^*)$ to be the map given by

$$e_J \mapsto \begin{cases} \pm e_I \wedge e_J & \text{if } I \cap J = \emptyset \\ 0 & \text{if } I \cap J \neq \emptyset \end{cases}$$

potentially correct sign!

hom.

Exercise: This is isom. in each degree, diagram commutes.

$\Rightarrow$ isom. between $Kos(\underline{x})$ and its dual.

The isom $H_i(\underline{x};M) \cong H^{n-i}(\underline{x};M)$ is then a trivial consequence. □

20/10/22

1.4. Regular Sequences

Now we get to "measuring how singular" a ring is - use Koszul cx. to study regular sequences.

Def 1.32 Let R be a ring and M an R-module.
An element $x \in R$ is *regular* of M if $xM \neq M$ and for any $m \in M$: $xm = 0 \Rightarrow m = 0$.
More generally, a sequence of elements $x_1, \dots, x_n$ is a *regular sequence* on M if:
- $(x_1, \dots, x_n)M \neq M$
- for each i, the element x_i is regular on $M/(x_1, \dots, x_{i-1})M$.

When $M = R$, we drop "on M" and say x is regular or $x_1, \dots, x_n$ is a regular sequence.

Rmk 1.33 (a) Suppose $(x_1, \dots, x_n)M \neq M$. Then x_i is regular on $M/(x_1, \dots, x_{i-1})M \iff ((x_1, \dots, x_{i-1}) :_M x_i) = (x_1, \dots, x_{i-1})M$.
(Def: $m \in ((x_1, \dots, x_{i-1}) :_M x_i)$ means $m \cdot x_i \in (x_1, \dots, x_{i-1})M$
x_i is reg on $M/(x_1, \dots, x_{i-1})M \iff (\overline{m} \cdot x_i = 0 \Rightarrow \overline{m} = 0) \iff m \in (x_1, \dots, x_{i-1})M$)

(b) When $(R, \mathfrak{m}, k)$ is noeth. local ring and $M \neq 0$ is fin. gen. R-module, then Nak. lemma gives us $(x_1, \dots, x_n)M \neq M$ automatically $\forall\, \underline{x} = x_1, \dots, x_n \in \mathfrak{m}$.

Example 1.34 (a) Consider $R = k[x_1, \dots, x_n]$ polynomial ring in n vars over a field k. The variable $x_1, \dots, x_n$ form a regular sequence on R. (x_i is NZD on $k[x_1, \dots, x_n]/(x_1, \dots, x_{i-1}) \cong k[x_i, \dots, x_n]$ — int. dom.)
(b) Let k be a field and $R = k[x, y, z]$. The sequence xy, xz is not regular on R since xz kills y on $R/(xy)$.
(c) Let $R = k[x, y]/(x^3 - y^2)$, k field. Then x, y is not reg. sequence:
(1) $(xy)R \neq R$ ($1 \notin (x, y)$). (2) x is NZD on R ($R \cong k[t^2, t^3]$), y is not NZD on $R/(x)$: $R/(x) \cong k[y]/(y^2)$

Note that the order of the elements in reg sequence matters:

Example 1.35 Let k be a field and $R = k[x, y, z]$. We claim that $x, (x-1)y, (x-1)z$ is a regular sequence, while $(x-1)y$,

$(x-1)z, x$ is not. (20)

For the first claim, note that over $R/(x)$, we have $(x-1)y = -y$ and $(x-1)z = -z$. Since $x, -y, -z$ is a reg. sequence on R, then so is $x, (x-1)y, (x-1)z$.

However, over $R/((x-1)y)$, the elements y and $(x-1)z$ are nonzero, but $(x-1)z \cdot y = \overline{0}$.

In particular, $(x-1)z$ is not regular on $R/((x-1)y)$ and $(x-1)y, (x-1)z, x$ is not a regular sequence.