Continue with Ext and Tor:

Rmk 1.12: Recall $\mathrm{Ext}^i_R(M,N)$ can be computed in the following way: Find a proj resolution of M

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

Apply the contravariant functor $\mathrm{Hom}_R(-,N)$ to this resolution. Get a (co-) complex:

$$0 \to \mathrm{Hom}_R(P_0,N) \to \mathrm{Hom}_R(P_1,N) \to \mathrm{Hom}_R(P_2,N) \to \cdots$$

Then its cohomology is $\mathrm{Ext}^*_R(M,N)$:

$$\mathrm{Ext}^i_R(M,N) = H^i\Big(0 \to \underset{0}{\mathrm{Hom}_R(P_0,N)} \xrightarrow{\delta_0} \underset{1}{\mathrm{Hom}_R(P_1,N)} \xrightarrow{\delta_1} \underset{2}{\mathrm{Hom}_R(P_2,N)} \to \cdots\Big)$$

$$\| \quad \ker(\delta_{i+1})/\mathrm{Im}(\delta_i) \qquad \text{e.g. } \mathrm{Ext}^0_R(M,N) = H^0(-\pi-) = \ker\big(\mathrm{Hom}_R(P_0,N) \to \mathrm{Hom}_R(P_1,N)\big)$$

$$P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\pi} M \to 0$$

$\ker\pi$, $f \searrow N$ with f, 0

$\ker\pi = \mathrm{Im}(\partial_1)$

Since $M \cong P_0/\ker\pi$, so get unique map $\bar{f}: P_0/\ker\pi \to N$

, apply $\mathrm{Hom}_R(-,N)$, get $\partial_1^*: \mathrm{Hom}_R(P_0,N) \to \mathrm{Hom}_R(P_1,N)$, $\partial_1^*(f) = f \circ \partial_1$

Then $f \in \ker(\partial_1^*) \Leftrightarrow f \circ \partial_1 = 0$. Since $\mathrm{Im}(\partial_1) = \ker\pi$

$\Rightarrow \ker(-\pi-) = \{f: P_0 \to N : f|_{\ker\pi} = 0\} \Leftrightarrow f(y) = 0 \ \forall y \in \ker\pi$

$\Rightarrow \ker(-\pi-) \cong \mathrm{Hom}_R(M,N)$

Alternatively, may compute Ext^1 with injective res. of M.

↑ W7

Similarly: $\mathrm{Tor}^R_*(M,N)$ can be computed via:

Find projective resolution: $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ of M.

Apply $-\otimes_R N$ to the resolution $\cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} 0$ (without M):

$$\cdots \to P_2 \otimes_R N \to P_1 \otimes_R N \to P_0 \otimes_R N \to 0.$$

Then its homology is $\mathrm{Tor}^R_i(M,N)$:

$$\mathrm{Tor}^R_i(M,N) = H_i(\cdots \to P_2 \otimes_R N \to P_1 \otimes_R N \to P_0 \otimes_R N \to 0)$$

Alternatively, do the same thing with proj. res. of N.

From this, we get some info on Ext/Tor from $\mathrm{pdim}_R(M) = n$ finite: $\mathrm{Tor}^R_i(M,-) = 0$ and $\mathrm{Ext}^i_R(M,-) = 0$, $\forall i > n$. (since Ext/Tor are computed from the proj. res's as above!)

Moreover, the ranks of the free modules in the minimal free resolution of M are important invariants of M:

Def 1.13 Let $F_\bullet$ be the min. free resolution of M. The *i-th Betti number* of M is $\beta_i(M) = \operatorname{rank}(F_i)$.

Rmk 1.14 Note that $\beta_0(M) = \dim_k(M/\mathfrak{m}M)$.

Example 1.15 Let $R = k[\![x,y]\!]$ and $M = R/(x^2, xy)$. Want to construct a min. free res. of M.

First note that M is cyclic, thus $R \to R/(x^2,xy) \to 0$ is start.
$1 \mapsto \bar{1}$

Relations on unique generator $\bar{1}$ are $x^2 \cdot 1 = 0$ and $xy \cdot 1 = 0$.

So proceed with

$$R^2 \xrightarrow{(x^2\ xy)} R \to R/(x^2,xy) \to 0$$

Now have to find relations between x^2 and xy on R, i.e. all $\alpha, \beta \in R$: $\alpha x^2 + \beta xy = 0 \iff \alpha x^2 = -\beta xy$

x is a NZD on R, so $\Rightarrow \alpha x = -\beta y$, this means $x \mid \beta$ and $y \mid \alpha$.

$\alpha = y\tilde{\alpha}$, $\beta = x\tilde{\beta}$, then $\tilde{\alpha} xy = -\tilde{\beta} xy \iff \tilde{\alpha} = -\tilde{\beta}$ and a relation is of the form $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} y \\ -\tilde{\alpha} x \end{pmatrix}$. Min. choice: $\tilde{\alpha} = 1$:

$y \cdot x^2 + (-x) \cdot xy = 0$ is a relation, and *all* other relations are multiples of this one. Thus:

$$R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{(x^2,\ xy)} R/(x^2,xy) \to 0 \quad \text{is exact.}$$

Now R is a domain, so $\begin{pmatrix} y \\ -x \end{pmatrix}$ is injective, and we are done!

$\Rightarrow$ free res. of M:

$$0 \to R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{(x^2\ xy)} R \to M \to 0$$

R in degree 2, R^2 in degree 1, R in degree 0.

This is indeed minimal (since we took min. # of generators in each step. Betti #s: $\beta_0(M) = 1$, $\beta_1(M) = 2$, $\beta_2(M) = 1$.

Moreover $\operatorname{pdim}_R(M) = 2$.

Ex 1.16 : $R=k[x,y,z]$, $M=R/(xy,xz,yz)$.

Claim : min free res :

$$0 \to R^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R^3 \xrightarrow{(xy,xz,yz)} R \to M \to 0$$

Again $\operatorname{pdim} M = 2$.

1.2. Resolution of the residue field

The residue field has in some sense the worst possible pdim:

Exercise Let (R,m,k) be a noeth. local ring, and M be a fin. gen. R-module. Show that $\beta_i(M) = \dim_k(\operatorname{Ext}^i_R(M,k)) = \dim_k(\operatorname{Tor}^R_i(M,k))$.

Rmk 1.17 As a consequence of this exercise, we get that

$$\operatorname{pdim}(M) = \sup\{i : \beta_i(M) \neq 0\} = \sup\{i : \operatorname{Ext}^i(M,k) \neq 0\} = \sup\{i : \operatorname{Tor}^R_i(M,k)^{*}\}$$

This can also be extended to the graded case, in part $\operatorname{Ext}^i(M,N)$ inherits a graded structure from a graded resolution of M, i.e. from graded structure of R.

Corollary 1.18 Let (R,m,k) be local/graded. Then for every fin gen. (graded) R-module M we have

$$\operatorname{pdim}_R M \leq \operatorname{pdim}_R(k).$$

Proof It suffices to consider the case when $\operatorname{pdim}_R(k)$ is finite. When $i > \operatorname{pdim}(k)$, we must have $\operatorname{Tor}^R_i(M,k) = 0$ (see Rmk 1.12) By Exercise above, $\beta_i(M) = 0$ for all such i, so $\operatorname{pdim}(M) \leq \operatorname{pdim}(k)$. □

Def 1.19 Let R be a domain with fraction field Q. The rank of a f. gen. R-module M is def as

$$\operatorname{rank} M := \dim_Q(M \otimes Q).$$

Ex 4 : Check that if M is a free module, then $\operatorname{rank} M$ is the free

Ex 5: • $\beta_i(M) = \operatorname{rank}(\Omega_i(M)) + \operatorname{rank}(\Omega_{i+1}(M))$ (rank of M)
• M f.g. over noeth. local domain, then $\sum_{i=0}^{\operatorname{pdim}(M)} (-1)^i \beta_i(M) = \operatorname{rk}(M)$.

→ Idea: If we have an explicit min res. of k, maybe we can say something about min. res. of other f.g. R-modules.

1.3. The Koszul complex (also see [Bruns-Herzog]) [Eisenbud]

One of the most important cx. in homological (commutative) algebra. Every sequence $x_1, \dots, x_n$ in any ring R gives rise to a Koszul complex.

Construction 1.20 (Koszul cx) The Koszul cx. on one element $x \in R$ is the complex $\operatorname{kos}(x) := 0 \to R \xrightarrow{\cdot x} R \to 0$ (degrees 1, 0).

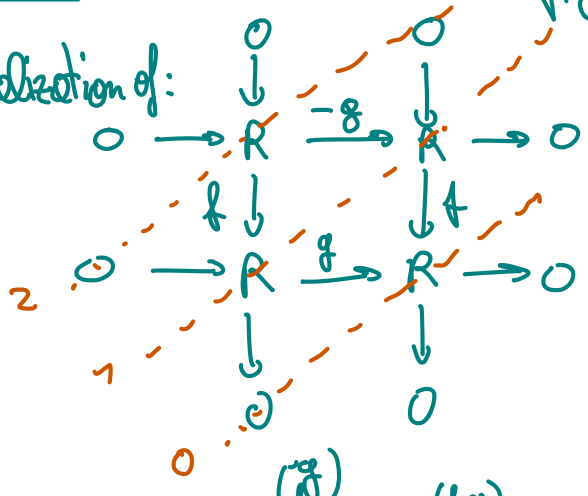
More generally: given $x_1, \dots, x_n \in R$, the Koszul cx w.r.t. $x_1, \dots, x_n$ is the complex $\operatorname{kos}(x_1, \dots, x_n)$ def. inductively as

$$\operatorname{kos}(x_1, \dots, x_n) := \operatorname{kos}(x_1, \dots, x_{n-1}) \otimes_R \operatorname{kos}(x_n).$$

There are different conventions for the Koszul cx. in the literature, but all lead to isom. cx's.

Example 1.21 The Koszul cx. on $f, g \in R$ is given by

$\operatorname{kos}(f,g)$ = totalization of:



which is $0 \to R \xrightarrow{\binom{g}{f}} R^2 \xrightarrow{(f\ g)} R \to 0$.

In general, computing succ. tensor products is tedious. Instead, use exterior algebra.

Def 1.22 The exterior algebra ΛM on an R-module M is obtained by taking the free R-algebra $R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \dots$ modulo the relations $x \otimes y = -y \otimes x$ and $x \otimes x = 0$ $\forall x, y \in M$. Denote the product on ΛM by $a \wedge b$.

ΛM is a graded alg. with homogeneous elements of deg d the image of $M^{\otimes d}$ (!).

ΛM is a skew-commutative algebra for all hom. elements a and b we have $a \wedge b = (-1)^{\deg(a)\cdot\deg(b)} b \wedge a$ and $a \wedge a = 0$ whenever a has odd degree. Denote the set of all deg n elements by $\Lambda^n M$.

This construction is functorial: a map $f: M \to N$ of R-modules induces a map $\Lambda M \xrightarrow{\Lambda f} \Lambda N$

$$m_1 \wedge \dots \wedge m_s \longmapsto f(m_1) \wedge \dots \wedge f(m_s)$$

We will mostly use this construction for free modules:

Denn $M = R^n$ with basis $e_1, \dots, e_n$, then $\forall\, 1 \le s \le n$

$\Lambda^s M = R^{\binom{n}{s}}$ with basis $e_{i_1} \wedge \dots \wedge e_{i_s}$ s.t. $i_1 < i_2 < \dots < i_s$.

Def 1.23 (Koszul cx with ext algebra) Let $x_1, \dots, x_n$ be elements in R.

The Koszul complex on $x_1, \dots, x_n$ is the complex

$$\mathrm{kos}(x_1, \dots, x_n) := 0 \to \Lambda^n R^n \to \Lambda^{n-1} R^n \to \dots \to \Lambda^1 R^n \to R \to 0$$

with differential $\partial(e_{i_1} \wedge \dots \wedge e_{i_s}) = \sum_{1 \le p \le s} (-1)^{p-1} x_{i_p} e_{i_1} \wedge \dots \wedge \widehat{e_{i_p}} \wedge \dots \wedge e_{i_s}$.

Ex 7 Show that ∂ def as above is indeed a differential.

Ex 8 Check that the 2 defs of Koszul cx. coincide.

Example 1.24 In the case of 2 elements, say x and y in R

$$\mathrm{kos}(x,y): \quad 0 \to \Lambda^2 R^2 \to \Lambda^1 R^2 \to R \to 0$$

with $\partial(e_1) = x$, $\partial(e_2) = y$, $\partial(e_1 \wedge e_2) = x e_2 - y e_1$, so

$$\mathrm{kos}(x,y) = 0 \to R \xrightarrow{\binom{-y}{x}} R^2 \xrightarrow{(x\ y)} R \to 0$$

Ex 9 : Write Koszul cx. for 3 elements f_1, f_2, f_3.

→ All def of Koszul cx. may have diff. signs conventions, but are isom. as cx.'s.

Def 1.25 Let M be a R-module and let $x_1, \dots, x_n \in R$. The Koszul complex on M w.r.t. $x_1, \dots, x_n$ is the complex

$$Kos(x_1, \dots, x_n; M) := Kos(x_1, \dots, x_n) \otimes_R M.$$

Rmk 1.26 In general, the Koszul cx. $Kos(x_1, \dots, x_n; M)$ looks like

$$0 \to M \to \underset{n}{M^n} \to \dots \to \underset{k}{M^{\binom{n}{k}}} \to \dots \to \underset{1}{M^n} \to \underset{0}{M} \to 0$$

and the nonzero entries in the diff. matrices consist of elements $x_1, \dots, x_n$ with carefully chosen signs. The leftmost map, in degree 1, is given as $\begin{pmatrix} x_1 \\ -x_2 \\ x_3 \\ \vdots \\ (-1)^{n+1} x_n \end{pmatrix}$ and the rightmost map is given as $(x_1 \; x_2 \cdots x_n)$.

This def. has the advantage that it has the obvious structure of a graded algebra. (Example of a DG algebra)

Also use the maps: functorial construction: given $\underline{x} = x_1, \dots, x_n \in R$ and $\underline{y} = y_1, \dots, y_m \in R$, any commutative diagram of R-module homomorphisms

$$\begin{array}{ccc} R^n & \xrightarrow{(x_1 \cdots x_n)} & R \\ \varphi_1 \downarrow & & \downarrow \varphi_0 \\ R^m & \xrightarrow[(y_1 \cdots y_m)]{} & R \end{array}$$

extends to a map of complexes $Kos(\underline{x}) \to Kos(\underline{y})$, by taking

$$\varphi_s(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \varphi_1(e_{i_1}) \wedge \cdots \wedge \varphi_1(e_{i_s}).$$

The fact that this map commutes with taking the differential is an immediate consequence of the definition, left as exercise.

Notation Given $x_1, \dots, x_n$ in a ring R, we often write $\underline{x} = x_1, \dots, x_n$.

Now get to the co-/homology of the Koszul cx:

Def 1.27 Let M be an R-module and $x_1, \dots, x_n \in R$. The i-th Koszul homology module of M on $x_1, \dots, x_n$ with coeffs in M, is the R-module

$$H_i(x_1, \dots, x_n; M) := H_i(Kos(x_1, \dots, x_n); M).$$

Exercise 1D: Let (R, m, k) be a noetherian local ring, and let $\underline{x}$ and $\underline{y}$ be two minimal generating sets for the same ideal. Show that $Kos(\underline{x})$ and $Kos(\underline{y})$ are isom. complexes.

Thm 1.28 Let R be a ring and $I = (x_1, \dots, x_n)$ be an ideal, and M be an R-module.

(1) $H_i(\underline{x}; M) = 0$ whenever $i < 0$ or $i > n$.

(2) $H_0(\underline{x}; M) = M/IM$.

(3) $H_n(\underline{x}; M) = (0 :_M I) = ann_M(I) = \{m \in M : m \cdot x = 0 \ \forall x \in I\}$
"$\{m \in M : x_1 m = \dots = x_n m = 0\}$" $= Hom_R(R/(x_1, \dots, x_n), M)$

(4) Every Koszul homology module $H_i(\underline{x}; M)$ is killed by $ann_R(M)$.

(5) Every Koszul homology module $H_i(\underline{x}; M)$ is killed by I.

(6) If M is a Noetherian R-module, so is $H_i(\underline{x}; M)$ $\forall i$.

(7) For every i, $H_i(\underline{x}; -)$ is a covariant additive functor $R\text{-Mod} \to R\text{-Mod}$.

(8) Every s.e.s. of R-modules

$$0 \to A \to B \to C \to 0$$

gives rise to a long exact sequence of homology,

$$\dots \to H_1(\underline{x}; C) \to H_0(\underline{x}; A) \to H_0(\underline{x}; B) \to H_0(\underline{x}; C) \to 0.$$

Proof (1) Immediate from the definition, because Koszul cx. is only nonzero in homological degrees 0 to n.