1.0. Setup

This course is mainly about commutative homological algebra.
Usually we will work in one of two settings:
(1) local setting
(2) graded setting

For this first some conventions:
All commutative rings are unital (with 1) and all ring homs send 1 to 1, moreover all rings are assumed to be noetherian.
For k a field, let $R = k[x_1, \dots, x_n]$ be the polyn. ring (sometimes later also denoted as S). R can be $\mathbb{N}$-graded, with $\deg(x_i) = d_i$ for some $d_i \in \mathbb{N}$. Most commonly: <u>standard grading</u>: $\deg(x_i) = 1 \ \forall i$.
Then can talk about homog. elements in R: e.g. with std grading in $\mathbb{Q}[x_1, x_2, x_3]$: $f = x_1x_2 + 3x_1^2 - \frac{8}{5}x_2^2$ is homog, $f = 1 + x_1x_2 + x_1^5$ is not.
Write $R_i := \{$ homog. elt. of R of deg $= i\}$.
R_i is an abelian group under $+$, and

$$R = \bigoplus_i R_i$$

Note that $R_i \cdot R_j \subseteq R_{i+j} \quad \forall i, j$
When R is std graded, we have

$$R_i = \bigoplus_{a_1 + \dots + a_n = i} x_1^{a_1} \cdots x_n^{a_n}.$$

More generally, a <u>graded ring</u> is a ring that can be decomposed into graded pieces $R = \bigoplus_i R_i$

and $R_i \cdot R_j \subseteq R_{i+j}$.

A graded R-module is then an R-module M s.t.

$$M = \bigoplus_i M_i \quad \text{and} \quad R_i M_j \subseteq M_{i+j}.$$

A hom. of graded R-modules $\varphi : M \to N$ which satisfies $\varphi(M_i) \subseteq N_{i+d}$ $\forall i$ is called a graded map of degree d.
A map of deg $= 0$ is called degree preserving.
One can shift degrees: write $M(-d)$ for the graded R-module that has M as the underlying R-mod., but where the graded structure is given as $M(-d)_i = M_{i-d}$.

Ex: $R = k[x,y]$ std grd.

	-1	0	1	2	3
R:	0	k	$kx \oplus ky$	...	

deg < 0 : 0
deg $= 0$: $R_0 = k$
deg $= 1$: $R_1 = kx \oplus ky$
⋮

$R(-1)$:

$R(-1)_0 = R_{0-1} = 0$
$R(-1)_1 = R_{1-1} = k$
$R(-1)_2 = R_{2-1} = kx \oplus ky$

	-1	0	1	2	3
	0	0	k	$kx \oplus ky$	

↖ $-d$ means: shift d-spaces to the right!

Note: $\deg(0) = -\infty$ (or other conventions!)
An ideal $I \subseteq R$ is homogeneous if it can be generated by homog. elements. This is equiv to

$$I = \bigoplus_i (I \cap R_i) \qquad (!)$$

If I is homog. and R is graded, then R/I is also nat. graded

$$(R/I)_i = R_i / I_i.$$

In most cases we work with fin. gen. $\mathbb{N}$-graded k-algebras R with $R_0 = k$, of the form $R = k[x_1, \dots, x_n]/I$, I homog.
A feature of this situation is that: $\exists$ only one homog.

max ideal, namely $R_+ := \bigoplus_{i>0} R_i$.

So get analogous behaviour to local rings (R, m).
BUT: always need to consider seperate proofs for both situations

So we have:

Local setting: R is local ring with (unique) max. ideal m.

Graded setting: $R = k[x_1, \dots, x_d]/I$, k a field, std grading, I homog, and $m = R_+$ is unique hom. max. ideal of R.

! In graded setting: all statements about graded mods and all homoms will be graded of deg 0.

1.1. Minimal free resolutions

M an R-module - how can we describe it?
Need to know generators + relations between generators + relations between relations + ... → construct free resolution of M
→ free resolutions tell us a lot about the module!

Def 1.1 Let M be a module over a ring. A projective resolution of M is a complex of projective R-modules F_i

$$F = \dots F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \dots \to F_1 \xrightarrow{\partial_1} F_0 \to 0$$

$$n \qquad n-1 \qquad 1 \qquad 0$$

s.t. $H_i(F) = 0 \ \forall i \neq 0$ and $H_0(F) \cong M$; α: $\partial_i \circ \partial_{i+1} = 0$ i.e. $\ker \partial_i \supseteq \operatorname{im}(\partial_{i+1})$

F_i proj: Def if: $P \to X \to U$ comm. $\exists P \to P$ for any surj R-alg $R \to X$ $\Rightarrow$ P is direct $\oplus$ of free R-mod

$$H_i(F) := \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$$

When all F_i are free, F is called a free resolution of M.
Usually: identify F with the augmented resolution, which is the exact sequence

$$F = \dots \to F_n \to \dots \to F_1 \to F_0 \to M \to 0$$

n — i.e. R noeth local — 0

Rmk: In the local setting, any proj. module must be free [Kaplansky 58] ← even inf. generated!
In graded setting: every bounded below graded proj. must be free.
⇒ In local/graded setting, any proj. res. is in fact free res!
"Free resolution = approximation of M by free modules"
Since every module is a quotient of a free module, every module has a free res.

Construction 1.3 (Minimal free res.)
Let M be a fin. gen. module over R, where R is either local or graded (→ setup!)
If M has β_0 many minimal generators, then we can write a surjective R-module homom.

$$R^{\beta_0} \xrightarrow{\pi_0} M.$$

If π_0 is an isom, then $M \cong R^{\beta_0}$ is a free module of rk $= \beta_0$.
Otherwise, $\ker(\pi_0) \neq 0$ is a submod of R^{β_0}, which must be f.gen, since R is noetherian.
If $\ker \pi_0$ is minimally generated by β_1 many elements we repeat the process and construct a surjective map

$$R^{\beta_1} \twoheadrightarrow \ker(\pi_0)$$

and compose it with $\ker(\pi_0) \hookrightarrow M$:

$$R^{\beta_0} \dashrightarrow R^{\beta_0} \xrightarrow{\pi_0} M \to 0$$

$$\ker(\pi_0) \quad (\to 0, \to 0)$$

Elts in $\ker(\pi_0)$ are called relations or syzygies of M on our chosen min. gens on M: if $m_1, \dots, m_{\beta_0}$ are the min. gens, then

$\pi_0 : R^{\beta_0} \to M : e_i \mapsto m_i$ and each $(r_1, \dots, r_{\beta_0}) \in \ker(\pi_0)$ corresponds to a relation on the m_i, meaning

$$r_1 m_1 + \dots + r_{\beta_0} m_{\beta_0} = 0.$$

$\ker(\pi_0)$ is called 1st syzygy module of M and is denoted by $\Omega_1(M)$.

Continue this process, construct a free resolution of M:

$$\dots \to F_n \xrightarrow{\pi_n} F_{n-1} \to \dots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \to 0.$$

In local/graded setting, we may choose # of gens minimal, so we can talk about minimal free resolutions.

Facts (Will not be proven):

- Every free resolution of M has a min. free res. of M as direct summand.
- Any two min. res's of M are isomorphic (as complexes) thus we can talk about the min. res. of M.
- $\Rightarrow$ the min. res. has the shortest length of any res. of M and M has a finite resolution $\Leftrightarrow$ the min. res. of M is finite.
- A free res. F of M is minimal with differential $\partial \iff \partial(F) \subseteq \mathfrak{m}F$ (i.e. $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$.). Thus, if we fix bases in the F_i, F_0 is minimal $\Leftrightarrow$ all entries in the matrices representing ∂ have all entries in $\mathfrak{m}$.

Def 1.4 In general, any ex. $(F_\cdot, \partial)$ with $\partial(F) \subseteq \mathfrak{m}F$ is called a minimal complex.

Ex: $R = k[x]$: $M = k \cong R/\langle x \rangle$

$$0 \to R \xrightarrow{\cdot x} R \xrightarrow{\pi_0} k \to 0 \qquad \cdot x \mapsto \langle x \rangle$$

$$F = \; R \xrightarrow[1]{\cdot x} R \;\; {}_0$$

$R = k[x,y]/(xy)$, $M = k[y] \cong R/(x)$

$$\to R \xrightarrow{\cdot y} R \xrightarrow{\cdot x} R \xrightarrow{\pi_0} \underset{M}{R/(x)} \to 0. \qquad \cdot y \mapsto \langle y \rangle, \; \cdot x \mapsto \langle x \rangle$$

∞ min. res.

$$F: \dots \to R \xrightarrow[2]{\cdot y} R \xrightarrow[1]{\cdot x} R \;\; {}_0$$

all min. $\partial_{2i} = \cdot y$, $\partial_{2i+1} = \cdot x$

Def 1.5 Let M be an R-module. A finite projective resolution $F_\bullet = \dots \to 0 \to F_c \to F_{c-1} \to \dots \to F_1 \to M_0 \to 0$ has **length** c if $F_c \neq 0$ and $F_i = 0$ $\forall i \geq c$.
A res. has infinite length if $F_i \neq 0$ $\forall i \geq 0$.
The **projective dimension** of M is

$$\mathrm{pdim}_R(M) := \inf\{c : M \text{ has a proj. res. of length } c\}.$$

Rmk 1.6 (cf Rmk 1.2) in the local/graded setting, any proj. res. is a free res. $\Rightarrow \mathrm{pdim}_R(M) = \inf\{c : M \text{ has a free res. of length } c\}$.

Note: If min. free res. is ∞, then set $\mathrm{pdim}_R(M) = \infty$.

Rmk 1.7 $\mathrm{pdim}_R(M) = 0 \iff M$ is projective (free in graded setting)
$\mathrm{pdim}_R(M) = 1 \iff \Omega_1(M)$ is projective

Def 1.8 Consider a min. free res. $F_\bullet$ of M. Then the **i-th syzygy module of M**, denoted $\Omega_i(M)$, is def. to be $\mathrm{Im}(\pi_i)$ or equiv. $\ker(\pi_{i-1})$.

Note that $\Omega_i(M)$ is def. only up to isom.

Rmk 1.9: Careful with notation: sometimes $\Omega_i(M) = \ker(\pi_i)$
→ Always check references!

Exercise 1: Show that for all R-modules M and $\forall i \geq 1$, if $F_\bullet$ is the min. free res. for M, then $\exists$ natural s.e.s.

$$0 \to \Omega_{i+1}(M) \to F_i \to \Omega_i(M) \to 0.$$

Rmk 1.10 If in Const. 1.3, we obtain at some pt an injective map of free modules, then its kernel is 0, so we have a finite free resolution.

Ex. 1.11 (∞ pdim of cyclic module): Let k be a field and

$R = k[x]/(x^2)$ with $m = (x)$, Then $k \cong R/(x)$ is cyclic with min. free res. $\cdots \to R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \cdots \xrightarrow{\cdot x} R \to k \to 0$

$\mathrm{pdim}_R(k) = \infty$.

Get also info on Ext and Tor: ← see later a brief intro of Ext + Tor!

Rmk 1.12: Recall $\mathrm{Ext}^i(M,N)$ can be computed in the following way: Find a proj. resolution of M

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

Apply the contravariant functor $\mathrm{Hom}_R(-,N)$ to this resolution. Get a (co-) complex:

$$0 \to \mathrm{Hom}_R(P_0,N) \to \mathrm{Hom}_R(P_1,N) \to \mathrm{Hom}_R(P_2,N) \to \cdots$$

Then its cohomology is $\mathrm{Ext}^*_R(M,N)$:

$$\mathrm{Ext}^i_R(M,N) = H^i\Big(0 \to \underset{0}{\mathrm{Hom}_R(P_0,N)} \xrightarrow{\partial_0^*} \underset{1}{\mathrm{Hom}_R(P_1,N)} \xrightarrow{\partial_1^*} \underset{2}{\mathrm{Hom}_R(P_2,N)} \to \cdots\Big)$$

$$\| \qquad \ker(\partial_{i+1}^*)/\mathrm{Im}(\partial_i^*)$$

e.g. $\mathrm{Ext}^0_R(M,N) = H^0(\cdots) = \ker\big(\mathrm{Hom}_R(P_0,N) \to \mathrm{Hom}_R(P_1,N)\big)$, ∂_1^* is the zero map

$P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\pi} M \to 0$

$\ker\pi$, $f \circ \partial_1 \overset{?}{=} 0$, f

$0 \to \ker\mathrm{Im}(\partial_1) = \ker\pi$

Since $M \cong P_0/\ker\pi$, so get unique map $\bar{f}: P_0/\ker\pi \to N$

, apply $\mathrm{Hom}_R(-,N)$, def $\partial_1^*: \mathrm{Hom}_R(P_0,N) \to \mathrm{Hom}_R(P_1,N)$, $\partial_1^*(f) = f \circ \partial_1$

Then $f \in \ker(\partial_1^*) \Leftrightarrow f \circ \partial_1 = 0$, Since $\mathrm{Im}(\partial_1) = \ker\pi$

$\Rightarrow \ker(-\circ\pi) = \{f: P_0 \to N : f|_{\ker\pi} = 0\} \Leftrightarrow f(y) = 0 \ \forall y \in \ker\pi$

$\Rightarrow \ker(-\circ\pi) \cong \mathrm{Hom}_R(M,N)$

Alternatively, may compute Ext^i with injective res. of M.