MATHM5253 EXERCISE SHEET 5 - THE LAST ONE!

DUE: MAY 1, 2018

Algebraic geometry, Gröbner bases

Problem 1. (a) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be two algebraic sets, and let

$$X \times Y = \{(x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{A}^{n+m} : (x_1, \dots, x_n) \in X \ (y_1, \dots, y_m) \in Y\}$$

be their Cartesian product. Show that $X \times Y$ is an algebraic set.

(b) Show that if both X and Y are irreducible, then also $X \times Y$ is irreducible.

Problem 2. (a) Show (by an example) that an infinite union of algebraic sets is not necessarily an algebraic set.

(b) Give an example of a maximal ideal J in $\mathbb{R}[x_1, \dots, x_n]$ such that $V(J) = \varnothing$. Why does this not contradict the Nullstellensatz?

Problem 3. (a) Show that the set $\{(x, 0) : x \neq 0, x \in \mathbb{R}\} \subset \mathbb{A}^2_{\mathbb{R}}$ is not an affine variety.

(b) Give an example to show that the set theoretic difference $X \backslash Y$ of two affine algebraic sets does not need to be an algebraic set.

Problem 4. Let $P_1, \dots, P_k \in K[x_1, \dots, x_n]$, I be an ideal in $K[x_1, \dots, x_n]$ and $<_\varepsilon$ be a monomial order on $\mathbb{N}^n$.

(a) Show that if $P_i \in I$ for all $i = 1, \dots, k$ and $\mathrm{lm}(I) = \langle \mathrm{lm}_\varepsilon(P_1), \dots, \mathrm{lm}_\varepsilon(P_k)\rangle$, then $P_1, \dots, P_k$ are a generating set for I.

(b) Let $\mathcal{P} = (P_1, \dots, P_k)$. Then show that $\overline{Q_1 + Q_2}^{\mathcal{P}} = \overline{Q_1}^{\mathcal{P}} + \overline{Q_2}^{\mathcal{P}}$ for all $Q_1, Q_2 \in K[x_1, \dots, x_n]$.

(c) If $\overline{Q_1}^{\mathcal{P}} = 0$ and $\overline{Q_2}^{\mathcal{P}} = 0$ and $A_1, A_2 \in K[x_1, \dots, x_n]$, then show that

$$\overline{A_1 Q_1 + A_2 Q_2}^{\mathcal{P}} = 0.$$

Problem 5. (a) Determine the cardinality of $V(f)$ where $f(z) = z^5 - z^4 + z^3 - 1$ is in $\mathbb{C}[z]$ and compare it to $\dim_{\mathbb{C}}(\mathbb{C}[z]/\langle z^5 - z^4 + z^3 - 1\rangle)$ (dimension here means vector space dimension).

(b) Same question for $V(x - 2y, y^2 - x^3 + x^2 + x)$ and $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/\langle x - 2y, y^2 - x^3 + x^2 + x\rangle$. Geometric interpretation?

(c) Same question for $V(x^3 - yz, y^2 - xz, z^2 - x^2 y)$ and $\dim_{\mathbb{C}}(\mathbb{C}[x, y, z]/\langle x^3 - yz, y^2 - xz, z^2 - x^2 y\rangle$. (Hint: Recall that $\dim_{\mathbb{C}}(\mathbb{C}[t]) = \infty$ and so also for any $\mathbb{C}$-module containing $\mathbb{C}[t]$)

Problem 6. (a) Fix a monomial order on $\mathbb{N}^3$ and let $K = \mathbb{C}$. Are the polynomials $P_1 = x^3 - yz$, $P_2 = x^2 y - z^3$ and $P_3 = y^2 - z^2$ a Gröbner basis with respect to this order?

(b) If not, then complete the polynomials to a Gröbner basis.

(c) Does the system of equations $P_1(x, y, z) = P_1(x, y, z) = P_2(x, y, z) = 0$ have a solution? (Try to answer this question without actaully calculating one!)