MATHM5253 EXERCISE SHEET 4

DUE: APRIL 16, 2018

Primary decomposition, Noether normalisation and the Nullstellensatz

Problem 1. (a) Let $I = \langle x^2, y^2, z^2 \rangle \cap \langle x+y \rangle \cap \langle x-y \rangle$ be an ideal in $R = \mathbb{R}[x,y,z]$. Is the given intersection of ideals a (minimal) primary decomposition of I? Explain!

(b) Let $R = K[x,y,z]$ be the polynomial ring over a field K, and let $\mathfrak{p}_1 = \langle x,y \rangle$, $\mathfrak{p}_2 = \langle x,z \rangle$, $\mathfrak{m} = \langle x,y,z \rangle$, and $J = \mathfrak{p}_1 \cdot \mathfrak{p}_2$ be ideals in R. Show that $J = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of J. Which components are isolated and which components are embedded?

Problem 2. (a) Show the proposition from the lecture: Let $R \subset S \subset T$ be rings. If S is a finite R-algebra and T is a finite S-algebra, then T is a finite R-algebra.

(b) Let $R = \mathbb{R}[x,y]/\langle x^5 - y^3 \rangle$. Show that $t = \frac{y}{x}$ and $u = \frac{x^2}{y}$ are integral over R. What are the R-module generators of $R[t]$ and $R[u]$?

Problem 3. Decompose $X := V((x^2y - xy^2)(x+y)) \subseteq \mathbb{A}^2_{\mathbb{R}}$ into irreducible components, that is, write X as a union of $V(f_i)$, where the f_i are irreducible polynomials. Same question for $X \subseteq \mathbb{A}^2_{\mathbb{F}_2}$, where $\mathbb{F}_2$ denotes the field with two elements.

Problem 4. Sketch the following affine algebraic sets (you may use a computer algebra program for this!)

(a) $V(y^2 - x^5) \subset \mathbb{A}^2_{\mathbb{R}}$
(b) $V((x^2+y^2)^2 + 4x(x^2+y^2) - 4y^2) \subset \mathbb{A}^2_{\mathbb{R}}$
(c) $V(x^2+y^2-1) \subset \mathbb{A}^3_{\mathbb{R}}$,
(d) $V(x^3 + x^2z^2 - y^2) \subset \mathbb{A}^3_{\mathbb{R}}$
(e) $V(x^4y^2 - x^2y^4 - x^4z^2 + y^4z^2 + x^2z^4 - y^2z^4) \subset \mathbb{A}^3_{\mathbb{R}}$

Problem 5. Let $F = (x^2 - y^3)^2 - (z^2 - y^2)^3$ be a polynomial in $\mathbb{R}[x,y,z]$.

(1) Sketch $V(F) \subset \mathbb{A}^3_{\mathbb{R}}$.
(2) Let $J_F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$ be the Jacobian ideal of F. Find $V(J_F)$ and sketch it.
(3) Is J_F radical?

Problem 6. The image of a non-constant complex polynomial map $f : \mathbb{C}^2 \to \mathbb{C}^3$ is a hypersurface. Let $f(s,t) = (s^3t^3, s^2, t^2)$.

(a) Find an irreducible polynomial map $F : \mathbb{C}^3 \to \mathbb{C}$ such that $\mathrm{Im}(f) \subset V(F)$. (Use coordinates (x,y,z).)

(b) Let again $J_F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$ be the Jacobian ideal of F. Find a minimal primary decomposition of J_F and its associated primes. (Hint: Ensure J_F is simplified as much as possible and try to guess the primary components!)

(c) Hence show that J_F has an embedded prime and two isolated primes.