MATHM5253 EXERCISE SHEET 3

DUE: MARCH 5, 2018

Radical, Modules, Nakayama and exact sequences

Problem 1. (a) Let $R = \mathbb{Q}[[x, y]]$ and let $J = \langle xy + y^3, x + x^2y, xy + 3y, x^4 - 5y^2 + x^2y \rangle$ be an ideal in R. Show that J is minimally generated by two elements in R.

(b) Let $R = K[t]$ and consider $M = K[t, t^{-1}]$ as R-module and let $I = tR$ be an ideal in R. Show that $M = IM$ but $M \neq 0$. Why does this example not contradict Nakayama's lemma?

Problem 2. Prove the isomorphism theorems for modules (without using the snake lemma).

Problem 3. (a) Let $0 \to A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \to 0$ and $0 \to B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \to 0$ be two short exact sequences of R-modules. Suppose that in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{\mu} & A & \xrightarrow{\varepsilon} & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{\mu'} & B & \xrightarrow{\varepsilon'} & B'' & \longrightarrow & 0 \end{array}$$

α', α'' are isomorphisms. Then show that α is an isomorphism too.

(b) Give an example of two short exact sequences $0 \to A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \to 0$ and $0 \to B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \to 0$ with $A' \cong B'$ and $A'' \cong B''$ but where A is not isomorphic to B. Why does your example not contradict (a)?

Problem 4. (Localisation of a module) Let R be a ring and $A \subset R$ be multiplicatively closed. Let M be an R-module.

(a) Show that $(m, a) \sim (n, b)$ if and only if $mbc = nac$ for some $c \in A$ defines an equivalence relation on $M \times A$.
(b) Writing $A^{-1}M$ for the set of equivalence classes of $\sim$, and $\frac{m}{a}$ for the class containing (m, a), show that the operation
$$\frac{m}{a} + \frac{n}{b} = \frac{bm + an}{ab}$$
is well defined and hence that $A^{-1}M$ is an abelian group.
(c) By defining an appropriate multiplication rule, show that $A^{-1}M$ has the structure of an $A^{-1}R$-module.

Problem 5. Let R be a ring and $A \subset R$ be multiplicatively closed.

(a) Suppose that $\phi : M \to N$ is a homomorphism of R modules. Show ϕ induces an $A^{-1}R$-homomorphism $A^{-1}M \to A^{-1}N$.
(b) Suppose $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules. Show that $0 \to A^{-1}L \to A^{-1}M \to A^{-1}N \to 0$, with the induced maps from (i), is an exact sequence of $A^{-1}R$-modules. (*Remark*: This means that localization is an exact functor from the category of R-modules to the category of $A^{-1}R$-modules)

Problem 6. Let R be a ring.

(a) Suppose that $R^m \cong R^n$. Show that $m = n$.
(b) Suppose that $\varphi : R^m \to R^n$ is surjective. Show that $m \geq n$.