MATHM5253 EXERCISE SHEET 1

DUE: FEBRUARY 5, 2018

Warm-up exercises

Problem 1. Work out the example from the first lecture: What are the integer solutions of $x^2 + y^2 = z^2$?
(1) First show that the rational solutions of $X^2 + Y^2 = 1$ are of the form

$$(X, Y) = \left(\frac{-2m}{1+m^2}, \frac{1-m^2}{1+m^2} \right), m \in \mathbb{Q}, \quad \text{and } (X, Y) = (0, 0) \, .$$

(2) From this find the integer solutions of the original equation.

Problem 2. Let $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with addition defined as $x \oplus y := \min(x, y)$ and multiplication $x \odot y := x + y$ for all $x, y \in \mathbb{R} \cup \{\infty\}$.

(a) Is $\mathbb{T}$ a commutative ring? If yes, then show that all axioms hold, if no, then explain which axiom fails.
(b) Calculate $3 \odot (5 \oplus 7)$, $(3 \oplus -3)^2$, and $(1 \oplus 8)^4$.
(c) Show that for any $x, y \in \mathbb{R} \cup \{\infty\}$, and any $k \in \mathbb{N}$, one has $(x \oplus y)^k = x^k \oplus y^k$.

Problem 3. (a) Prove that if $\varphi : R \to S$ is a ring isomorphism then $\varphi^{-1} : S \to R$ is a ring homomorphism, and hence also an isomorphism.
(b) Let R be a ring and $I \subseteq R$ be an ideal and let $\varphi : R \to R/I$ be the canonical projection. Show that $\ker \varphi = I$ and φ is a ring homomorphism.

Problem 4. Let I, J and K be ideals of a ring R. Show that

(a) $I \cap J$ and IJ are ideals
(b) $IJ \neq I \cap J$,
(c) $I(J + K) = IJ + IK$,

Problem 5. Let I, J and K be ideals of a ring R. Recall that $(I : J) = \{r \in R : rJ \subset I\}$. Show that

(a) $(I : J)$ is an ideal of R and $I \subset (I : J)$,
(b) $J \subset I$ implies that $(I : J) = R$,
(c) $IJ \subset K$ if and only if $I \subset (K : J)$.

Problem 6. Let R be a commutative ring and let $I, J \subseteq R$ be ideals.

(a) Let $\sqrt{I} = \{r \in R : r^n \in I \text{ for some positive integer } n\}$. Show that $\sqrt{I}$ is an ideal that contains I. [Note: $\sqrt{I}$ is called the *radical of* I.]
(b) Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
(c) Let $R = k[x, y]$. Show that $\sqrt{(x^2, y^2)} = (x, y)$ and that $\sqrt{(x^2) \cap (y^2)} = (xy)$.