MAT 412 HOMEWORK 9

DUE: MARCH 24, 2017 (BEGINNING OF CLASS)

This homework set covers sections 6.2, 6.3. References are to Hungerford, 3rd. edition.

Problem 1.

(The second isomorphism theorem): Let I and J be ideals in a ring R. Prove that

$$I/(I \cap J) \cong (I+J)/J\,.$$

Problem 2. (a) Let I and J be ideals in a ring R such that $J \subseteq I$. Prove that $I/J = \{a + J : a \in I\}$ is an ideal in the quotient ring R/J.

(b) (The third isomorphism theorem): Let I, J, R be as in (a). Show that

$$(R/J)/(I/J) \cong R/I\,.$$

(c) Show that $\mathbb{Z}_5 \cong \mathbb{Z}_{15}/(5)\mathbb{Z}_{15}$.

Problem 3. (a) Show that $\mathbb{R}[x,y]/(x^3 - y^2)$ is isomorphic to $\mathbb{R}[t^2, t^3]$. [Hint: First homomorphism theorem. First show that $f(x,y) = x^3 - y^2$ is in the kernel of the map φ as defined in the lecture. In order to see that $(f(x,y))$ is the full kernel, you may use the fact, that the kernel of φ is generated by elements of the form $x^a y^b - x^{a'} y^{b'}$, where $a, a', b, b' \in \mathbb{N}$. This fact can be proved using Gröbner bases methods, which is beyond the scope of our course!]

(b) Is $(x^3 - y^2)$ a prime ideal in $\mathbb{R}[x,y]$? Explain!

Problem 4. (a) Let $f(x), g(x) \in K[x]$, K a field. Show that the ideal $(f(x), g(x)) = (d(x))$, where $d(x)$ is the gcd of $f(x)$ and $g(x)$.

(b) Use part (a) to show that every ideal in $K[x]$ is principal.

(c) Show that the ideal $(x^4 - 5x^3 + 7x^2 - 5x + 6, x^4 + 2x^2 + 1, x^4 - 2x^3 + x^2 - 2x)$ in $\mathbb{R}[x]$ is maximal.

Problem 5. (a) Prove that a commutative ring R with identity has a unique maximal ideal if and only if the set of nonunits in R is an ideal. (6.3.B.19) You may use the fact that every ideal of R except R itself is contained in a maximal ideal. [This is known as *Krull's theorem*] Such a ring R is called a *local ring*.

(b) Recall the definition of $R = K[[t]]$, the power series ring in one variable: $K[[t]]$ is the set of all power series $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots$, with $a_i \in K$. In HW 6 you showed that R is a ring. Now show that R is a local ring.

(c) Show that the ideal generated by t and the ideal generated by $e^t - 1$ in $\mathbb{R}[[t]]$ are equal.

Problem 6. (a) Let $f : R \to S$ be a homomorphism of commutative rings with $f(1_R) = 1_S$. Show that if $\mathfrak{p}$ is a prime ideal in S, then its preimage $f^{-1}(\mathfrak{p})$ in R is also a prime ideal. [Hint: First isomorphism theorem]

(b) Is the statement also true if you replace "prime" by "maximal"? Explain!

Problem 7. Read sections 7.1, 7.2, 7.3 in the book.