MAT 412 HOMEWORK 8

DUE: MARCH 17, 2017 (BEGINNING OF CLASS)

This homework set covers sections 5.3, 6.1. References are to Hungerford, 3rd. edition.

Problem 1. (a) Show that $\mathbb{Z}_2[x]/(x^3+x+1)$ is a field that contains all 3 roots of x^3+x+1. (5.3.B.9)

(b) Show that the fields $\mathbb{Q}[\sqrt{5}]$ and $\mathbb{Q}[\sqrt{2}]$ are not isomorphic. [Note: You may find problem 4 from HW 7 useful.]

Problem 2. Let R be a commutative ring with identity and let $I, J \subseteq R$ be ideals.

(a) Let $\sqrt{I} = \{r \in R : r^n \in I$ for some positive integer $n\}$. Show that $\sqrt{I}$ is an ideal that contains I. (6.1.B.38) [Note: $\sqrt{I}$ is called the *radical of* I.]

(b) Let $I \cap J = \{x \in R : x \in I$ and $x \in J\}$. Show that $I \cap J$ is an ideal. Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

(c) Let $R = k[x,y]$. Show that $\sqrt{(x^2, y^2)} = (x,y)$ and that $\sqrt{(x^2) \cap (y^2)} = (xy)$.

Problem 3. Let R be a commutative ring with identity and let $I, J \subseteq R$ be ideals.

(a) Show that $IJ = \{a_1b_1 + \cdots + a_nb_n : n \geq 1, a_k \in I, b_k \in J\}$,i.e., the set of all possible finite sums of elements of the form ab, $a \in I$, $b \in J$, is an ideal in R. [6.1.B.36]

(b) Show that $I : J := \{r \in R : rJ \subset I\}$ is an ideal in R and that $I \subseteq I : J$. [Note: the ideal $I : J$ is called *colon ideal* or sometimes *ideal quotient*.]

(c) Let $R = K[x,y,z]$. Calculate $(xz : yz) : (z)$, $(x, yz) : (x,y)$, and $(x,y) : (x)$.

Problem 4. (a) Let K be a field. Let $X \subset K^n$ be any subset of the affine n-space K^n. Describe $I(X) = \{f \in K[x_1, \ldots, x_n] : f(p) = 0$ for all $p \in X\}$ as a subset of the polynomial ring, i.e., show that $I(X)$ is an ideal in $K[x_1, \ldots, x_n]$.

(b) Let $X = \{(0,0,0)\}$ in $\mathbb{R}^3$. Find $I(X)$.

(c) Let $I = (xy - x^3) \subseteq \mathbb{R}[x,y]$. Find a set $X \in \mathbb{R}^2$ such that $I = I(X)$.

Problem 5. (a) Let $X \subset K^n$ be as above. Show that $I(X)$ is a radical ideal, i.e., $\sqrt{I(X)} = I(X)$.

(b) Let $X, Y \subset K^n$ be any subsets. Show that if $X \subset Y$, then $I(X) \supset I(Y)$.

(c) Let $X, Y \subset K^n$ be any subsets. Show that $I(X \cup Y) = I(X) \cap I(Y)$.

Problem 6. Let $I, J \subseteq K[x_1, \ldots, x_n]$ be ideals, and denote by $V(I)$, $V(J)$ their zerosets in K^n (i.e., $V(I) = \{(a_1, \ldots, a_n) \in K^n : f(a_1, \ldots, a_n) = 0$ for all $f \in I\}$ and similarly $V(J)$).

(a) Prove that $V(I \cdot J) = V(I \cap J) = V(I) \cup V(J)$. [Hint: Prove the inclusions: $V(I) \cup V(J) \subseteq V(I \cap J)$, $V(I \cap J) \subseteq V(I \cdot J)$, $V(I \cdot J) \subseteq V(I) \cup V(J)$.]

(b) Let $J = (x^2y, x^2z, xyz, xz^2, xy^2, y^2z, yz^2) \subseteq \mathbb{R}[x,y,z]$. Show that $V(J)$ is the union of the three coordinate lines in $\mathbb{R}^3$. Is J a radical ideal, i.e., $J = \sqrt{J}$? If not, find an ideal $\mathcal{K}$ such that $\mathcal{K} = I(V(\mathcal{K}))$. [Hint: part (a) of this problem and 2 (b)]

Problem 7. Read sections 6.2, 6.3, 7.1 in the book.