MAT 412 HOMEWORK 7

DUE: MARCH 10, 2017 (BEGINNING OF CLASS)

This homework set covers sections 4.6, 5.1 and 5.2. References are to Hungerford, 3rd. edition.

Problem 1. (a) Factor the polynomials $x^2 + x - 2$ and $x^3 - x^2 + 2x - 2$ in $\mathbb{R}[x]$.
(b) Find two polynomials $f(x), g(x) \in \mathbb{R}[x]$ such that $[f(x)], [g(x)] \neq [0]$ but $[f(x)][g(x)] = [0]$ in $\mathbb{R}[x]/(x^3 - x^2 + 2x - 2)$.
(c) Show that $\mathbb{R}[x]/(x^2 + x - 2)$ is not a field. [*Hint:* it is enough to show that there are zerodivisors in this ring!]

Problem 2. Let K be a field, and let $p(x)$ be a nonconstant polynomial in $K[x]$. Show that $K[x]/(p(x)^n)$ for $n \geq 2$ is not an integral domain.

Problem 3. Let K be a field and $p(x, y) \in K[x, y]$, where $K[x, y]$ is the polynomial ring in 2 variables and assume that $p(x, y)$ is not a constant polynomial. Define a relation $\equiv \pmod{p(x, y)}$ on $K[x, y]$ similar as for the polynomial ring in one variable.

(a) Show that your $\equiv \pmod{p(x, y)}$ on $K[x, y]$ is an equivalence relation.
(b) Describe the elements of $K[x, y]/(y)$.
(c) Consider $\mathbb{R}[x, y]/(xy)$. First describe the elements in this set [Note: later we will see that $\mathbb{R}[x, y]/(xy)$ is a ring!] and then find a bijection to $(\mathbb{R}[x] \times \mathbb{R}[y]) \backslash \{(f(x), g(y)) | f(0) \neq g(0)\}$.

Problem 4. (a) Show that the ring $\mathbb{Q}[\sqrt{2}]$ consisting of all elements of the form $a + b\sqrt{2}$, $a, b \in \mathbb{Q}$ is isomorphic to $\mathbb{Q}[x]/(x^2 - 2)$.
(b) Show that $\mathbb{Q}[\sqrt{2}]$ is a field. Is $\mathbb{Q}[\pi]$ also a field? [*Hint:* You may find it useful to consult Problems 5,6 from Homework 6.]

Problem 5. (a) Show that, under congruence modulo $p(x) = x^3 + 2x + 1$ in $\mathbb{Z}_3[x]$ there are exactly 27 distinct congruence classes. (5.1.A.4)
(b) In general, for $f(x) \in \mathbb{Z}_p[x]$, p prime and $\deg f(x) = n \geq 1$, how many distinct congruence calsses are there modulo $f(x)$? Explain your answer!

Problem 6. (a) Write out the addition and multiplication tables for $R = \mathbb{Z}_2[x]/(x^3 + x + 1)$. Is R an integral domain or even a field?
(b) Write out the addition and multiplication tables for $R = \mathbb{Z}_2[x]/(x^3 + 1)$. Is R an integral domain or even a field?

Problem 7. Read sections 5.3, 6.1 and 6.2 in the book.