MAT 412 HOMEWORK 6

DUE: FEBRUARY 17, 2017 (BEGINNING OF CLASS)

This homework set covers sections 4.1, 4.2, 4.3. References are to Hungerford, 3rd. edition.

Problem 1. Give examples for the following or explain why the objects don't exist:

(a) Polynomials $f(x), g(x) \in M_2(\mathbb{R})[x]$ such that $\deg(f(x)g(x)) < \deg f(x) + \deg g(x)$.
(b) Polynomials $f(x), g(x) \in \mathbb{Z}_{2017}[x]$ such that $\deg(f(x)g(x)) < \deg f(x) + \deg g(x)$.
(c) A polynomial $f(x) \in \mathbb{Z}_9[x]$ of degree ≥ 1 that is a unit in $\mathbb{Z}_9[x]$.

Problem 2. Let R be a ring. The set of all *formal power series* $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots$, with $a_i \in R$ forms a ring which is denoted by $R[[t]]$. [Note: There is no requirement of convergence of $f(t)$!]

(a) If $R = \mathbb{Q}$ find an element $f(t)$ of $\mathbb{Q}[[t]]$ that is not a polynomial.
(b) Prove that the formal power series $R[[t]]$ form a ring.
(c) Prove that a formal power series $f(t)$ is invertible if and only if a_0 is a unit in R.

Problem 3. (a) Calculate $\gcd(x^3 - ix^2 + 4x - 4i, x^2 + 1)$ in $\mathbb{C}[x]$. (4.2.A.5)
(b) Calculate $\gcd(4x^2 - 1, 2x^3 + x^2 - 4x - 2)$ in $\mathbb{Z}_2[x]$. Does the $\gcd(4x^2 - 1, 2x^3 + x^2 - 4x - 2)$ also exist in $\mathbb{Z}[x]$?

Problem 4. Find all irreducible polynomials in $\mathbb{Z}_2[x]$ of degree ≤ 4. Same question for $\mathbb{Z}_3[x]$.

Problem 5. Let $K \subseteq L$ be an inclusion of fields and $a \in L \backslash K$. Define $K[a]$ to be the set of all elements of L of the form

$$c_0 + c_1 a + c_2 a^2 + \cdots + c_n a^n ,$$

where $n \geq 0$ and all $c_i \in K$.

(a) For a concrete example: For $K = \mathbb{Q}$, $L = \mathbb{R}$ and $a = \pi$, show that $\mathbb{Q}[\pi]$ is a subring of $\mathbb{R}$.
(b) Show in general that $K[a]$ is a subring of L.

Problem 6. Notation as in Problem 5.

(a) Show that the evaluation map from the polynomial ring $\varphi_a : K[x] \to K[a], f(x) \mapsto f(a)$ is a surjective ringhomomorphism (written diferently: $\varphi_a(f(x)) = f(a)$).
(b) Again for $K = \mathbb{Q}$ and $L = \mathbb{R}$, show that $\mathbb{Q}[\pi]$ is isomorphic to $\mathbb{Q}[x]$. For this you may use the fact that $\sum_{i=0}^{n} c_i \pi^i = 0$ with $c_i \in \mathbb{Q}$, if and only if each $c_i = 0$. [We won't prove this fact!]
(c) Is $\mathbb{Q}[\sqrt{2}]$ isomorphic to $\mathbb{Q}[x]$?

Problem 7. Read appendix G and sections 4.4, 4.5 and 4.6 in the book.