# MAT 412 HOMEWORK 5

DUE: FEBRUARY 10, 2017 (BEGINNING OF CLASS)

This homework set covers sections 3.2, 3.3. References are to Hungerford, 3rd. edition.

**Problem 1.** (a) Show that $R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{R}$. Is $R$ a field?
(b) Find a unit in $R$, which is $\neq \pm 1$.
(c) Show that $K = \mathbb{Q}[\sqrt{-5}]$ is a field.

**Problem 2.** An element $e$ in a ring $R$ is called *idempotent* if $e^2 = e$.

(a) Find all idempotents in $\mathbb{Z}$ and $\mathbb{Z}_{12}$. (3.2.A.3)
(b) Assume that $R$ has an identity and that for every $r \in R$ one has $r^2 = r$, that is, every $r$ is idempotent. Show that $R$ is a commutative ring.

**Problem 3.** A *semiring* is a set $S$ together with two binary operations $\oplus$ and $\odot$ such that $\oplus$ satisfies $(ASS), (COM), (NEU)$ and $\odot$ satisfies $(ASS), (NEU)$ and $\oplus, \odot$ satisfy $(DIST)$:

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c), \text{ and } (a \oplus b) \odot c = (a \odot c) \oplus (b \odot c) \text{ for all } a, b, c \in S.$$

Moreover, the multiplication with $0_S$ annihilates $S$, that is $0_S \odot s = s \odot 0_S = 0_S$ for all $s \in S$. Let $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with addition defined as $x \oplus y := \min(x, y)$ and multiplication $x \odot y := x + y$ for all $x, y \in \mathbb{R} \cup \{\infty\}$.

(a) Show that $\mathbb{T}$ is a commutative semiring.
(b) Calculate $3 \odot (5 \oplus 7)$, $(3 \oplus -3)^2$, and $(1 \oplus 8)^4$.
(c) Show that for any $x, y \in \mathbb{R} \cup \{\infty\}$, and any $k \in \mathbb{N}$, one has $(x \oplus y)^k = x^k \oplus y^k$.

**Problem 4.** Let $R$ be ring with identity of characteristic $n > 0$.

(a) Prove that $nr = 0_R$ for any $r \in R$. (3.2.B.43)
(b) If $R$ is an integral domain, prove that $n$ is prime.
(c) If $R$ is a commutative integral domain of characteristic $p > 0$, then show that for any $a, b \in R$, the freshman's dream holds true, that is: $(a + b)^p = a^p + b^p$.
(d) For $R$ as in (c), show that the *Frobenius map* $F : R \to R, r \mapsto r^p$ is a ring homomorphism.

**Problem 5.** (a) Let $f : R \to S$ be an epimorphism of rings and assume that $R$ has a unit $1_R$. Show that $f(1_R)$ is the unit element of $S$.
(b) Let $\alpha : \mathbb{R} \to \mathbb{R}$ be the map $x \mapsto 3x$. Let $\beta : \mathbb{Z} \to \mathbb{Z}^2$ be the map $x \mapsto (x, 0)$ (here consider $\mathbb{Z}^2$ as a ring with componentwise addition and multiplication). Are $\alpha, \beta$ homomorphisms? If yes, are they mono-, epi-, or isomorphisms?

**Problem 6.** Let $f : R \to S$ be a ring homomorphism. Show that $\ker f$ (as defined in class) is a subring of $R$. Is $\ker f$ also closed under multiplication in $R$, i.e., is $r \cdot_R x \in \ker f$ for any $r \in R$ and any $x \in \ker f$? Is $\operatorname{im} f$ closed under multiplication in $S$?

**Problem 7.** Read ahead sections 4.1, 4.2 and 4.3 in the book.