MAT 412 HOMEWORK 4

DUE: FEBRUARY 3, 2017 (BEGINNING OF CLASS)

This homework set covers section 2.2, 2.3 and 3.1. References are to Hungerford, 3rd. edition.

Problem 1. (a) Write out the addition and multiplication table of $\mathbb{Z}_4$ and $\mathbb{Z}_6$.
(b) Write out the addition and multiplication table of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ (Here addition and multiplication is defined *componentwise*, i.e., $([a],[b]) \oplus ([c],[d]) = ([a] \oplus [c], [b] \oplus [d])$ and $([a],[b]) \odot ([c],[d]) = ([a] \odot [c], [b] \odot [d])$)
(c) If you compare the tables for $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, are they the "same"? Same question for $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$.

Problem 2. (2.2.B.14) Solve the following equations:

(a) $x^2 + x = [0]$ in $\mathbb{Z}_3$
(b) $x^2 + x = [0]$ in $\mathbb{Z}_6$.
(c) If p is prime, show that the only solutions of $x^2 + x = [0]$ in $\mathbb{Z}_p$ are $[0]$ and $[p-1]$.

Problem 3. (a) Find all the units and zero-divisors in $\mathbb{Z}_8$ and $\mathbb{Z}_{19}$.
(b) How many units are there in $\mathbb{Z}_p$, where p is prime?
(c) Do all elements of $\mathbb{Z}_8$ have square-roots? (An element $[a]$ in $\mathbb{Z}_n$ is a square-root if there exists a $[b] \in \mathbb{Z}_n$ such that $[a] \odot [a] = [b]$).

Problem 4. Show that every nonzero element of $\mathbb{Z}_n$ is either a unit or a zero divisor, but not both. (2.3.B.10)

Problem 5. (a) Show that $\mathbb{R}^2$ together with componentwise addition and multiplication $*$ defined by

$$(a,b) * (c,d) = (ac - bd, ad + bc)$$

is a commutative ring. We will write R' for this ring, that is, R' is the ring $(\mathbb{R}^2, +, *)$. [To find the multiplicative unit $1_{R'}$ it might be useful to find a solution for the equation $(a,0) * (x,y) = (a,0)$]
(b) Show that in R', the equation $x^2 + 1_{R'} = 0_{R'}$ has a solution. [*Remark:* In fact, one can show that every polynomial equation has a solution in R']
(c) The ring R' is better known under which name?

Problem 6. Let $M_2(\mathbb{R})$ be the set of all 2×2-matrices with real entries. Define $\mathrm{GL}_2(\mathbb{R}) = \{A \in M_2(\mathbb{R})$: there exists $B \in M_2(\mathbb{R})$, such that $AB = BA = \mathbb{1}_2\}$.
(a) Is $\mathrm{GL}_2(\mathbb{R})$ a ring with matrix addition and matrix multiplication? If not, which properties fail?
(b) Let $D := \{A \in M_2(\mathbb{R}) : A$ is a diagonal matrix, i.e., $A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\}$. Is D a ring with matrix addition and matrix multiplication? Are there zero-divisors in D? (A zero-divisor in D would be an $A \neq 0_D \in D$ such that there exists a $B \in D$ with $AB = 0_D$).

Problem 7. Read ahead sections 3.2 and 3.3 in the book.