MAT 412 HOMEWORK 3

DUE: JANUARY 27, 2017 (BEGINNING OF CLASS)

This homework set covers section 2.1 and 14.1 (the Chinese Remainder Theorem). References are to Hungerford, 3rd. edition.

Problem 1. Consider the following relation $\sim$ on $\mathbb{R}^2$: two points (x_1, x_2) and (y_1, y_2) are related, that is, $(x_1, x_2) \sim (y_1, y_2)$, if there exists a $c \in \mathbb{R}\backslash\{0\}$ such that $x_1 = cy_1$ and $x_2 = cy_2$.
(a) Show that $\sim$ is an equivalence relation and describe its equivalence classes.
(b) What happens if we consider the relation $\sim'$ with $(x_1, x_2) \sim' (y_1, y_2)$ if there exists a $c \in \mathbb{R}$ such that $x_1 = cy_1$ and $x_2 = cy_2$? Is $\sim'$ still an equivalence relation? If not, which property fails?

Problem 2. Let k be any integer.

(a) If $[k] = [1]$ in $\mathbb{Z}_n$, prove that $(k, n) = 1$. Is the converse true as well? (2.1.B.16)
(b) What is the congruence class of $(2k + 1)^2$ modulo 8? What is the congruence class of $(2k)^2$ modulo 8?
(c) Use (b) to prove or disprove the following statement: There exist integers u and v such that $u^2 + v^2 = 2019$.

Problem 3. Let $n \in \mathbb{N}$ be any natural number and $p \in \mathbb{N}$ be prime.

(a) Show that $n^{p-1} \equiv 1 (\mathrm{mod}\, p)$ for the two following pairs: $n = 2, p = 5$; $n = 4, p = 7$; (cf. 2.1.A.1)
(b) Show that this holds in general, i.e., show that $n^p \equiv n (\mathrm{mod}\, p)$. [Hint: Use induction on n. You may also find Problem 5 (c) from HW sheet 2 useful.]
(c) The result in (b) is well-known under which name?

Problem 4. (a) If $u \equiv v (\mathrm{mod}\, n)$ and u is a solution of $6x + 5 \equiv 7 (\mathrm{mod}\, n)$, then show that v is also a solution. (14.1.A.1)
(b) If $6x + 5 \equiv 7 (\mathrm{mod}\, n)$ has a solution, show that one of the numbers $1, 2, 3, \ldots, n - 1$ is also a solution. (14.2.A.1)

Problem 5. Use the Chinese remainder theorem to solve the following systems of congruences: (14.1.A.8&10)

(a) $x \equiv 5 (\mathrm{mod}\, 6)$, $x \equiv 7 (\mathrm{mod}\, 11)$.
(b) $x \equiv 1 (\mathrm{mod}\, 2)$, $x \equiv 2 (\mathrm{mod}\, 3)$, $x \equiv 3 (\mathrm{mod}\, 5)$.

Problem 6.

Do there exist 1000 consecutive integers that are not square-free? (Here n is *square-free* if there does not exist a $k \neq \pm 1 \in \mathbb{Z}$ such that $k^2 | n$) [Hint: Chinese Remainder Theorem]

Problem 7. Read ahead sections 2.2, 2.3 and 3.1 in the book.