MAT 412 HOMEWORK 11

DUE: APRIL 7, 2017 (BEGINNING OF CLASS)

This homework set covers sections 7.3, 7.4, 7.5. References are to Hungerford, 3rd. edition.

Problem 1. Let G be a group and let $\Delta = \{(g, g) : g \in G\}$ be the "diagonal". Prove that Δ is a subgroup of $G \times G$ and that this group is isomorphic to G.

Problem 2. (a) Find a group isomorphism between an additive and a multiplicative group, and then find another group homomorphism between such groups that is not an isomorphism.
(b) In the first case, determine the inverse map, and in the second case the kernel and image.

Problem 3. (a) Find the order of the smallest subgroup of A_5 that contains the permutation $(12)(34)$.
(b) Write (12534) as a product of transpositions.

Problem 4. (a) Compute the sign of the permutations $\sigma = (143)$ and $\tau = (23)(412)$ in S_4.
(b) If σ, τ are elements of S_n, check that $\sigma\tau\sigma^{-1}\tau^{-1}$ always lies in A_n, and that $\tau\sigma\tau^{-1}$ belongs to A_n whenever σ is an even permutation.

Problem 5. Let G be a group and X be a set. A map $h : G \times X \to X$ is called a (left) *group action* if it satisfies the two conditions:
(GA1) $h(e_G, x) = x$ for all $x \in X$,
(GA2) $h(ab, x) = h(a, h(b, x))$ for all $a, b \in G$ and for all $x \in X$ (written for short as: $(ab)x = a(bx)$).

(a) Show that $h : G \times X \to X$ is a group action if and only if the induced map $\tilde{h} : G \to \mathrm{Bij}(X, X), a \mapsto (x \mapsto h(a, x))$ is a group homomorphism. (Here $\mathrm{Bij}(X, X)$ denotes the set of bijections $X \to X$).
(b) Show that for $X = \mathbb{R}^n$, $G = \mathrm{GL}_n(\mathbb{R})$, the map $h : \mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, (A, v) \mapsto A \cdot v$ is a left action.

Problem 6. Let $G \times X \to X$ be a left group action. Then for any $x \in X$, the set $G \cdot x = \{gx, g \in G\} \subseteq X$ is called the *orbit of* x and $G_x = \{g \in G : gx = x\} \subseteq G$ is called the *stabilizer* of x.

(a) Show that G_x is a subgroup of G.
(b) Show that $(\mathbb{R}^*, \cdot) \times \mathbb{R}^2 \to \mathbb{R}^2, (t, x, y) \mapsto (tx, t^{-1}y)$ is a left group action and describe the orbits and stabilizers. [There are essentially three types of orbits and stabilizers: for points (x, y) with $x \neq 0$ and $y \neq 0$; one of the coordinates is 0; and $(x, y) = (0, 0)$!]
(c) Show that $\rho : S_n \times \mathbb{R}^n \to \mathbb{R}^n$ acts by permuting coordinates of a vector $v \in \mathbb{R}^n$. Find two non-trivial subspaces that stay invariant under the action of S_n (a subspace $W \subseteq \mathbb{R}^n$ is invariant under the action if for any $w \in W : \rho(\sigma, w) \in W$ for all $\sigma \in S_n$).

Problem 7. Read sections 8.1, 8.2, 8.3, 8.4 in the book.