MAT 412   HOMEWORK 10

DUE: MARCH 31, 2017 (BEGINNING OF CLASS)

This homework set covers sections 7.1, 7.2, 7.3. References are to Hungerford, 3rd. edition.

Problem 1. Let G be the set of all 4-th roots of unity in $\mathbb{C}$, i.e., $G = \{\zeta \in \mathbb{C} : \zeta^4 = 1\}$.

(a) Show that G forms an abelian group with the natural operation.
(b) Compare G and $(\mathbb{Z}_4, +)$
(c) Compute the sum of all elements in G.

Problem 2. Find an operation $*$ that makes $X = \{(A, b) \in \mathbb{Q}^{n^2} \times \mathbb{Q} : \det(A) \cdot b = 1\}$ into a group. Is $(X, *)$ abelian? Explain!

Problem 3. (a) Show that a subset $H \subseteq G$ of a group G is a subgroup if and only if for all $x, y \in H$: $xy^{-1} \in H$.
(b) Find three finite and two infinite subgroups of SO_3.
(c) Find four subgroups of S_4, the group of permutations on four elements.

Problem 4. (a) Consider a regular pentagon P. Write down the operation table for the group of rotational symmetries of P [Hint: Ex. 5 in 7.1 in the book].
(b) List the elements of D_5, the full group of symmetries of P (that is, also consider reflections) (7.1.22) [Hint: Show that $D_5 = \{e, \sigma, \sigma^2, \sigma^3, \sigma^4, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3, \tau\sigma^4\}$ for a suitable rotation σ and a reflection τ.]
(c) Consider the group D_5 as a subgroup of $\mathrm{GL}_3(\mathbb{R})$. Is D_5 also a subgroup of SO_3? [Hint: Consider the pentagon in the xy-plane and centered in the origin of $\mathbb{R}^3$. You have to write elements in D_5 as 3×3 matrices. First find the five matrices describing the rotations in D_5. Then find the matrix for the reflection and use part (b) to find the remaining four matrices.]

Problem 5. (a) Let $G = \{e, a, b\}$. Show that there is only one way to make G into a group of order 3. Is this an abelian group?
(b) Let $K = \{e, a, b, c\}$. Find two different multiplications on K that make K into a group of order 4. Are both of your groups abelian?

Problem 6. (a) Show that the center of $\mathrm{GL}_2(K)$,where K is a field, is

$$Z(\mathrm{GL}_2(K)) = \{A \in \mathrm{GL}_2(K) : A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \neq 0\}.$$

(b) Generalize the statement to $\mathrm{GL}_n(K)$. [Hint: All elements in $Z(\mathrm{GL}_n(K))$ commute with any element M in $\mathrm{GL}_n(K)$. Look at the matrices $M = \mathbb{1}_n + E_{ij}$, where E_{ij} is the matrix that has a 1 at the ij-th spot and zeros everywhere else.]

Problem 7. Read sections 7.4, 7.5, 8.1 in the book.