MATH 3195/5195 Notes

Thursday, March 19

Recall: For an ideal $J \subseteq K[x_1,\dots,x_n]$ we defined the (K is always a field!) vanishing set $V(J) = \{ \cancel{f \in K[x_1,\dots,x_n]} : p \in \mathbb{A}^n_K : f(p)=0 \ \forall f \in J\}$

And for any set $X \subseteq \mathbb{A}^n_K$ we have the vanishing ideal

$$I(X) = \{ f \in K[x_1,\dots,x_n] : f(x) = 0 \ \forall x \in X \}.$$

Some properties of $I(-)$:

Prop (1) $I(\emptyset) = K[x_1,\dots,x_n]$. If K is infinite, then $I(\mathbb{A}^n_K) = \{0\}$.
(see ex. below, why $|K| = \infty$ is necessary!)

(2) $X \subseteq Y \subseteq \mathbb{A}^n_K \Rightarrow I(Y) \subseteq I(X)$

(3) $X, Y \subseteq \mathbb{A}^n_K \Rightarrow I(X \cup Y) = I(X) \cap I(Y)$.

Pf: (1) Definition: $I(\emptyset) = \{ f \in K[x_1,\dots,x_n] : \underbrace{f(x_1,\dots,x_n) = 0 \ \forall x \in \emptyset}_{\text{no condition on } f} \}$

$= K[x_1,\dots,x_n]$.

Second part follows from a lemma for the proof of NN (see §17 → Lemma 17.3. in lecture notes).
If $|K| = \infty$, $\Rightarrow \forall f \in K[x_1,\dots,x_n] : f \neq 0$, then $\exists (\alpha_1,\dots,\alpha_n) \in \mathbb{A}^n_K$ s.t. $f(\alpha_1,\dots,\alpha_n) \neq 0$.

(2) Again use def - exercise! [If $f \in I(Y) \Rightarrow f(y) = 0 \ \forall y \in Y$
in part: $f(x) = 0 \ \forall x \in X \subseteq Y$
$\Rightarrow f \in I(X)$]

(3) Here $f \in I(X \cup Y) \Leftrightarrow f(x) = 0 \ \forall x \in X \cup Y \Leftrightarrow f(x) = 0 \ \forall x \in X \ \forall x \in Y$
$\Leftrightarrow f \in I(X)$ and $f \in I(Y) \Leftrightarrow f \in I(X) \cap I(Y)$. □

example (in (1): need $|K| = \infty$) Set $K = \mathbb{F}_p = \mathbb{Z}_p$ for $p \in \mathbb{Z}$ prime.
Then $f(x) = x^p - x \in K[x_1,\dots,x_n]$ vanishes $\forall x \in \mathbb{A}^1_{\mathbb{F}_p} \Rightarrow 0 \neq f \in I(\mathbb{A}^1_{\mathbb{F}_p})$.

Now we want to study the composition of the two maps I and V. One of them is "easy", for the other we need the Nullstellensatz:

<u>Prop</u> Let K be a field, $J \subseteq K[x_1, \dots, x_n]$ an ideal and $X \subseteq \mathbb{A}^n_K$ a subset. Then:

(i) $X \subseteq V(I(X))$, with $=$ if and only if X is algebraic.

(ii) $J \subseteq I(V(J))$.
[ex for $\subsetneq$]

<u>Ex</u>: (i) Let $X = \mathbb{Z} \subseteq \mathbb{R} = \mathbb{A}^1_{\mathbb{R}}$. We have seen $I(\mathbb{Z}) = \langle 0 \rangle$

$\Rightarrow V(I(\mathbb{Z})) = V(\langle 0 \rangle) = \mathbb{A}^1_{\mathbb{R}} \neq \mathbb{Z}$

(ii) [ex for $=$ and $\subsetneq$] For $J = (x^2 - y^2) \subseteq \mathbb{A}^2_{\mathbb{R}}$

$=$: $I(V(J)) = I(V(x-y) \cup V(x+y)) = I(V(x-y)) \cap I(V(x+y))$
$= \langle x-y \rangle \cap \langle x+y \rangle = \langle x^2 - y^2 \rangle = J$.

$\subsetneq$: $J = \langle x^2 \rangle \subseteq K[x]$. But $I(V(\langle x^2 \rangle)) = I(V(\langle x \rangle)) = I(\{0\}) = \langle x \rangle \supsetneq J$

Moreover even: In $\mathbb{R}[x]$ for $J = \langle x^2 + 1 \rangle$

$I(V(J)) = I(\emptyset) = \mathbb{R}[x] \rightsquigarrow J \subsetneq \mathbb{R}[x]$.

<u>Pf of Prop</u> Inclusions: tautological, use defs.(!)

[$\because$ (i) $x \in X \Rightarrow$ If $f(x) = 0$, i.e. $f \in I(X)$, then $x \in V(I(X))$.
(ii) $f \in J \Rightarrow \forall a \in V(J)$: $f(a) = 0 \Rightarrow f \in I(V(J))$]

(i) [statement about $=$]

$\Rightarrow$: Assume $X = V(I(X))$, then X is algebraic, since $X = V(J)$ for the ideal $J = I(X) \subseteq K[x_1, \dots, x_n]$.

$\Leftarrow$: Assume X algebraic. Then $X = V(J)$ for some ideal $J \subseteq K[x]$
But then $J \subseteq I(X)$ and thus $V(I(X)) \subseteq V(J) = X$.

[Recall: K is alg. closed if every non-constant poly in $K[x]$ has a root in K].

Now: what about $\supseteq$ in (ii)?

Thm (Hilbert's Nullstellensatz – geometric version) Let K be algebraically closed field and let $J \subseteq K[x_1,\dots,x_n]$ be an ideal.

Then:

(weak form) $J \neq K[x_1,\dots,x_n] \Rightarrow V(J) \neq \emptyset$.

(strong form) $I(V(J)) = \sqrt{J}$.

We will prove this thm in §17. [Level 5 only].

Rmk This thm says that we have correspondences:

$$\begin{array}{ccc}
\{\text{radical ideals} \subseteq K[x_1,\dots,x_n]\} & \longleftrightarrow & \{\text{alg. subsets } X \subseteq \mathbb{A}^n_K\} \\
\cup & & \cup \\
\{\text{prime ideals}\} & \longleftrightarrow & ? \\
\cup & & \cup \\
\{\text{max. ideals}\} & \longleftrightarrow & \{\text{points } p \in \mathbb{A}^n_K\}
\end{array}$$

weak alg. form of HNS.

The correspondence: $\{\text{prime ids}\} \longleftrightarrow ?$ will be tightly connected to primary decomp.

Ex (i) Let $p = \langle x^2 - y^3 z^3 \rangle \subseteq \mathbb{C}[x,y,z]$ prime

Pic in $\mathbb{R}^3$:  "Limão" $\rightsquigarrow$ "irreducible", 1 component

(ii) Let $J = \langle x^2 - y^2 \rangle \subseteq K[x,y]$. Have seen:

$J = \langle x-y \rangle \cap \langle x+y \rangle$ not prime

$V(J) = V(x-y) \cup V(x+y)$

2 components.

(3) Let $J = \langle x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, y^2z, yz^2, xyz\rangle \subseteq K[x,y,z]$

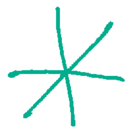
Have seen (in sec §14), that

$$J = \langle x,y\rangle \cap \langle x,z\rangle \cap \langle y,z\rangle \cap \langle x^2,y^2,z^2,xyz\rangle \text{ is min. P.D.}$$

$$\sqrt{J} = \langle x,y\rangle \cap \langle x,z\rangle \cap \langle y,z\rangle$$

$$\Rightarrow V(J) = V(\sqrt{J}) = V(x,y) \cup V(x,z) \cup V(y,z)$$

union of 3 coord.-axes ⟷ isolated primes in PD of J ⟷ irred. comp. of $V(J)$!



Lemma Every non-empty set of alg. subsets of $\mathbb{A}^n_K$ has a minimal element.

Pf: Exercise (in lecture notes!).

Def An algebraic set $X \subseteq \mathbb{A}^n_K$ is irreducible if, whenever $X = X_1 \cup X_2$ for $X_i \subseteq \mathbb{A}^n_K$ algebraic, then either $X = X_1$ or $X = X_2$.

In some textbooks: irreducible algebraic set = alg. variety

For us: alg variety and algebraic set are synonymous.

Prop (i) Let $X \subseteq \mathbb{A}^n_K$ be an algebraic set and $I(X)$ be the vanishing ideal of X. Then X is irreducible $\Leftrightarrow$ $I(X)$ is prime in $K[x_1,\dots,x_n]$

(ii) Any algebraic set has an expression

$$X = X_1 \cup \dots \cup X_m$$

unique up to permutation of the X_i, with X_i irreducible and $X_i \nsubseteq X_j$ for $i \neq j$. The X_i are called the irreducible components of X.

Pf: For (i) prove: X is reducible $\Leftrightarrow$ $I(X)$ is not prime.

$\Rightarrow$: Let $X = X_1 \cup X_2$ be a nontrivial decomposition into alg. sets. Then $X_i \subsetneq X$ means that:

$\exists f_1 \in I(X_1) \setminus I(X)$ and $\exists f_2 \in I(X_2) \setminus I(X)$.

But then $(f_1 \cdot f_2)(x) = 0 \quad \forall x \in X$, i.e. $f_1 \cdot f_2 \in I(X)$

$\xrightarrow[\text{prime id}]{\text{def of}}$ $I(X)$ is not prime.

$\Leftarrow$: If $I(X)$ is not prime, then. $f_1, f_2 \notin I(X)$ s.t. $f_1 \cdot f_2 \in I(X)$

Set $X_i = V(I(X) + \langle f_i \rangle) \quad i=1,2.$

Then by Prop above: $X_i = V(I(X)) \cap V(\langle f_i \rangle)$

$[V(I+J) = V(I) \cap V(J)]$ $\quad = X \cap V(\langle f_i \rangle)$ (since X is alg)

$\subsetneq X$, since $f_i \notin I(X)$.

$\Rightarrow X_1 \cup X_2 \subseteq X$.

Moreover: $(I(X) + \langle f_1 \rangle)(I(X) + \langle f_2 \rangle) = \underbrace{I(X)^2 + \langle f_1 \rangle I(X) + \langle f_2 \rangle I(X) + \langle f_1 f_2 \rangle}_{\subseteq I(X)}$

$\supseteq I(X)$

Since: $X = V(I(X))$ (by Prop 16.11 (i))

and $V(I(X)) \subseteq V((I(X) + \langle f_1 \rangle)(I(X) + \langle f_2 \rangle)) = X_1 \cup X_2$.

$[I \subseteq J \Rightarrow V(J) \subseteq V(I)]$

$\Rightarrow X \subseteq X_1 \cup X_2$. But $X_i \subsetneq X$, so X is reducible.

(ii) Let Σ be the set of alg subsets of $\mathbb{A}^n_k$, which do not have such a decomposition. If $\Sigma = \emptyset$, then we are done

(16.14)

Otherwise, by Lemma above, there is a minimal element $X \in \Sigma$. If X is irred, then $X \notin \Sigma$. ↯.
Otherwise X has a nontrivial decomp $X = X_1 \cup X_2$, but since X is minimal, it follows that $X_i \notin \Sigma$, $i = 1,2$.
Thus X_i both have a decomposition into irreducibles.
But then $X_1 \cup X_2 = X$ has a decomp. into irreducibles. ↯
$\Rightarrow \Sigma = \emptyset$ and thus existence ✓. □

Rmk $I(X)$ is a radical ideal and the decomposition of X into irreducibles X_i corresponds to a minimal primary decomposition of $I(X)$. (Note: $I(X)$ does not have embedded components)
Then: $\mathrm{Ass}(I(X)) \longleftrightarrow I(X_i)$ $\quad X = X_1 \cup \dots \cup X_m$

ex (1) Let $X = V(\underbrace{\langle xz, yz \rangle}_{J}) \subseteq \mathbb{A}^3_{\mathbb{C}}$.

$I(X) = \langle xz, yz \rangle$, since $\langle xz, yz \rangle$ is radical and strong HNS:
$\cap\!\!\mid$
$\mathbb{C}[x,y,z]$ $\quad I(V(J)) = \sqrt{J}$.

What are irred. comp. of X? ⇝ primary dec. of $I(X) =$
$\overbrace{\langle xz, yz \rangle}^{I} = \langle z \rangle \cap \langle x, y \rangle$

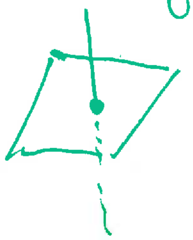
$\boxed{\text{eg use}}$: $\sqrt{\underbrace{I : \langle z \rangle}_{\langle x,y \rangle}} = \sqrt{\langle x, y \rangle} = \langle x, y \rangle$

$\sqrt{I : \langle x \rangle} = \sqrt{\{\alpha \in R : \alpha x \in \langle xz, yz \rangle\}} = \sqrt{\langle z \rangle} = \langle z \rangle$

$\Rightarrow \langle xz, yz \rangle \overset{!}{=} \langle z \rangle \cap \langle x, y \rangle$
$\supseteq$: Let $\alpha x + \beta y = \gamma z \Rightarrow z \mid \alpha y + \beta x$, only possible if $z \mid \alpha$ and $z \mid \beta$
$\subseteq$: clear

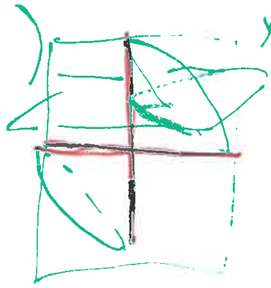
Now $V(\langle x,y\rangle \cap \langle z\rangle) = V(\langle x,y\rangle) \cup V(\langle z\rangle)$

<u>Pic</u> in $\mathbb{R}^3$



$X = \{z=0\text{-plane}\} \cup \{z\text{-axis}\}$

(2) See lecture notes for calculations: $V(\underbrace{x^2-yz}_{f_1}, \underbrace{xz-x}_{x(z-1)})$



Start with analyzing the 2 eqns:

$\{x^2-yz=0\} \rightsquigarrow$ cone irreducible

$\{xz-x=0\}$
$\quad \overset{\shortparallel}{\underset{f_2}{x}}\underbrace{(z-1)}_{f_3} = 0$ 2 planes $\rightsquigarrow V(\langle f_1, f_2 f_3\rangle) = V(\langle f_1\rangle + \langle f_2 f_3\rangle)$

$= V(\langle f_1\rangle) \cap V(\langle f_2 f_3\rangle)$

$= V(\langle f_1\rangle) \cap [V(\langle f_2\rangle) \cup V(\langle f_3\rangle)]$

$= [V(\langle f_1\rangle) \cap V(\langle f_2\rangle)] \cup [V(\langle f_1\rangle) \cap V(\langle f_3\rangle)]$

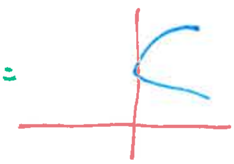
$\boxed{\text{cone} \cap \text{plane } x=0}$

$V(yz, x) = 2$ lines $\{x=y=0\} \cup \{z=x=0\}$

$V(x^2-yz, z-1)$
$= V(x^2-y, z-1)$

$\boxed{\text{"parabola in } \{z=1\}\text{-plane"}}$

Here $V(x^2-yz, xz-x) =$



$X = V(x,y) \cup V(x,z) \cup V(x^2-y, z-1)$

↳ This Explanation see lecture notes!

(3) Let $J = \langle y^2, xy \rangle \in k[x,y]$

$J = \langle y \rangle \cap \langle y^2, x \rangle$ is min. primary decomp.

$\Rightarrow V(J) = V(\langle y \rangle \cap \langle y^2, x \rangle) = V(\langle y \rangle) \cup V(\langle y^2, x \rangle)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \quad V(\langle x, y \rangle)$

$= \underbrace{\{(x,y) : y = 0\}}_{\text{isolated}} \cup \underbrace{\{(0,0)\}}_{\text{embedded pt.}}$

To sum up, get the dictionary:

Algebra		Geometry
• radical ideal		alg. variety = alg set
J	$\longrightarrow$	$V(J)$
$I(X)$	$\longleftarrow$	X
• sum of ideals		intersection of varieties
$J_1 + J_2$	$\longrightarrow$	$V(J_1) \cap V(J_2)$
$\sqrt{I(X) + I(Y)}$	$\longleftarrow$	$X \cap Y$
(mult) • intersection of ideals		union of varieties
$I \cap J \quad (I \cdot J)$	$\longrightarrow$	$V(I) \cup V(J) \; (= V(I \cdot J))$
$I(X) \cap I(Y) \quad (\sqrt{I(X) \cdot I(Y)})$	$\longleftarrow$	$X \cup Y$
can show: • quotients		difference of varieties
$I : J$	$\longrightarrow$	$\overline{V(I) \setminus V(J)}$ ← closure $\{a \in \mathbb{A}^n_K : f(a) = 0 \ \forall f \in I : J\}$
$I(X) : I(Y)$	$\longleftarrow$	$\overline{X \setminus Y}$
• prime ideal	$\longleftrightarrow$	irred variety
• min prim. decomp		decomp into irred. comp.
$I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$	$\longrightarrow$	$V(I) = V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_m)$
$I(X) = \bigcap_{i=1}^{m} I(X_i)$	$\longleftarrow$	$X = \bigcup_{i=1}^{m} X_i$
• max ideal	$\longleftrightarrow$	points in $\mathbb{A}^n_K$
• ACC on ideals	$\longleftrightarrow$	DCC on varieties