

MATH3195/5195M EXERCISE SHEET 4

DUE: MARCH 18, 2020

Primary decomposition, Noether normalisation and the Nullstellensatz

**Some quick questions**

- (1) Let  $R = K[x, y, z]$  and  $I = \langle x^3, y^2, z \rangle$ . Show that  $I$  is a primary ideal. Are  $J = \langle x - y^3 - z^2 \rangle$  and  $L = \langle x + 1, y - 1, z + 1 \rangle$  primary ideals? Explain!
- (2) Show that  $f = x^3 + x$  is integral over  $K[x^2]$ . Show that  $K[x^2] \subseteq K[x]$  is an integral extension by finding an integral dependence relation for any  $f \in K[x]$ .
- (3) Show that  $\frac{x}{y}$  is integral over  $K[x, y]/\langle x^2 - y^3 \rangle$ .

- Problem 1.** (a) Let  $I = \langle x^2, y^2, z^2 \rangle \cap \langle x + y \rangle \cap \langle x - y \rangle$  be an ideal in  $R = \mathbb{R}[x, y, z]$ . Is the given intersection of ideals a (minimal) primary decomposition of  $I$ ? Explain!
- (b) Let  $I$  be a *monomial ideal* in  $\mathbb{Q}[x_1, \dots, x_n]$ , that is,  $I$  is generated by monomials. Show that if  $R$  is generated by pure powers of a subset of the variables, then it is a primary ideal. Further show that if  $r = r_1 r_2$  is a minimal generator of  $I$ , where  $r_1$  and  $r_2$  are relatively prime, then

$$I = (I + \langle r_1 \rangle) \cap (I + \langle r_2 \rangle).$$

*Remark:* This yields an algorithm to compute primary decomposition of a monomial ideal!

- Problem 2.** (a) Let  $R = \mathbb{R}[x, y]/\langle x^5 - y^3 \rangle$ . Show that  $t = \frac{y}{x}$  and  $u = \frac{x^2}{y}$  are integral over  $R$ . What are the  $R$ -module generators of  $R[t]$  and  $R[u]$ ?
- (b) Let  $f = x^3 + y^2$ . Show that  $f$  is integral over  $\mathbb{Q}[x^6, y^2]$ .
- (c) Let  $R$  be a unique factorisation domain, that is,  $R$  is an integral domain and every element in  $R$  can be written as a product of irreducible elements, unique up to order and multiplication with units. Show that every integral element of the form  $\frac{x}{y}$ ,  $x, y \in R$  is already contained in  $R$ . (*Remark:* this shows that  $R$  is *integrally closed* in its field of fractions).

**Problem 3.** Decompose  $X := V((x^2y - xy^2)(x + y)) \subseteq \mathbb{A}_{\mathbb{R}}^2$  into irreducible components, that is, write  $X$  as a union of  $V(f_i)$ , where the  $f_i$  are irreducible polynomials. Same question for  $X \subseteq \mathbb{A}_{\mathbb{F}_2}^2$ , where  $\mathbb{F}_2$  denotes the field with two elements.

**Problem 4.** Sketch the following affine algebraic sets (you may use a computer algebra program for this!)

- (a)  $V(y^2 - x^5) \subset \mathbb{A}_{\mathbb{R}}^2$
- (b)  $V((x^2 + y^2)^2 + 4x(x^2 + y^2) - 4y^2) \subset \mathbb{A}_{\mathbb{R}}^2$
- (c)  $V(x^2 + y^2 - 1) \subset \mathbb{A}_{\mathbb{R}}^3$ ,
- (d)  $V(x^3 + x^2z^2 - y^2) \subset \mathbb{A}_{\mathbb{R}}^3$
- (e)  $V(x^4y^2 - x^2y^4 - x^4z^2 + y^4z^2 + x^2z^4 - y^2z^4) \subset \mathbb{A}_{\mathbb{R}}^3$

**Problem 5.** Let  $F = (x^2 - y^3)^2 - (z^2 - y^2)^3$  be a polynomial in  $\mathbb{R}[x, y, z]$ .

- (1) Sketch  $V(F) \subset \mathbb{A}_{\mathbb{R}}^3$ .
- (2) Let  $J_F = \langle \partial_x(F), \partial_y(F), \partial_z(F) \rangle$  be the Jacobian ideal of  $F$ . Find  $V(J_F)$  and sketch it.
- (3) Is  $J_F$  radical?

**Problem 6.** The image of a non-constant complex polynomial map  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^3$  is a hypersurface. Let  $f(s, t) = (s^3 t^3, s^2, t^2)$ .

- (a) Find an irreducible polynomial map  $F : \mathbb{C}^3 \rightarrow \mathbb{C}$  such that  $\text{Im}(f) \subset V(F)$ . (Use coordinates  $(x, y, z)$ .)
- (b) Let again  $J_F = \langle \partial_x(F), \partial_y(F), \partial_z(F) \rangle$  be the Jacobian ideal of  $F$ . Find a minimal primary decomposition of  $J_F$  and its associated primes. (Hint: Ensure  $J_F$  is simplified as much as possible and try to guess the primary components!)
- (c) Hence show that  $J_F$  has an embedded prime and two isolated primes.