MATH3195/M5195 EXERCISE SHEET 3

DUE: MARCH 8, 2020

Radical, Modules, Nakayama and exact sequences

Problem 1. Let R be a ring and consider $R[[x]]$, the ring of formal power series with coefficients in R (an element $f \in R[[x]]$ is of the form $f = \sum_{n=0}^{\infty} a_n x^n$ with $a_n \in R$). Show the following:

(a) f is a unit in $R[[x]]$ if and only if a_0 is a unit in R.
(b) $f \in J(R[[x]])$ if and only if $a_0 \in J(R)$.
(c) Let K be a field. Then $K[[x]]$ is a local ring with maximal ideal (x). (One can also show that $K[[x_1, \ldots, x_n]]$ is a local ring with maximal ideal $(x_1, \ldots, x_n)$).

Problem 2. (a) Let $R = \mathbb{Q}[[x, y]]$ and let $J = \langle xy + y^3, x + x^2y, xy + 3y, x^4 - 5y^2 + x^2y \rangle$ be an ideal in R. Show that J is minimally generated by two elements in R.
(b) Let $R = K[t]$ and consider $M = K[t, t^{-1}]$ as R-module and let $I = tR$ be an ideal in R. Show that $M = IM$ but $M \neq 0$. Why does this example not contradict Nakayama's lemma?

Problem 3. Prove the isomorphism theorems for modules (without using the snake lemma).

Problem 4. (a) Prove the 3×3-lemma: Let R be a ring. Assume that

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\alpha} & A_2 & \xrightarrow{\alpha'} & A_3 & \longrightarrow & 0 \\
 & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \\
0 & \longrightarrow & B_1 & \xrightarrow{\beta} & B_2 & \xrightarrow{\beta'} & B_3 & \longrightarrow & 0 \\
 & & {\scriptstyle g_1}\downarrow & & {\scriptstyle g_2}\downarrow & & {\scriptstyle g_3}\downarrow & & \\
0 & \longrightarrow & C_1 & \xrightarrow{\gamma} & C_2 & \xrightarrow{\gamma'} & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

is a commutative diagram of R-modules and all columns and the middle row is exact. Show that the top row is exact if and only if the bottom row is exact.

(b) Give an example of two short exact sequences $0 \to A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \to 0$ and $0 \to B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \to 0$ with $A' \cong B'$ and $A'' \cong B''$ but where A is not isomorphic to B.

Problem 5. (Localisation of a module) Let R be a ring and $A \subset R$ be multiplicatively closed. Let M be an R-module.

(a) Show that $(m, a) \sim (n, b)$ if and only if $mbc = nac$ for some $c \in A$ defines an equivalence relation on $M \times A$.

(b) Writing $A^{-1}M$ for the set of equivalence classes of $\sim$, and $\frac{m}{a}$ for the class containing (m, a), show that the operation

$$\frac{m}{a} + \frac{n}{b} = \frac{bm + an}{ab}$$

is well defined and hence that $A^{-1}M$ is an abelian group.

(c) By defining an appropriate multiplication rule, show that $A^{-1}M$ has the structure of an $A^{-1}R$-module.

Problem 6. Let R be a ring and $A \subset R$ be multiplicatively closed.

(a) Suppose that $\phi : M \to N$ is a homomorphism of R modules. Show ϕ induces an $A^{-1}R$-homomorphism $A^{-1}M \to A^{-1}N$.

(b) Suppose $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules. Show that $0 \to A^{-1}L \to A^{-1}M \to A^{-1}N \to 0$, with the induced maps from (i), is an exact sequence of $A^{-1}R$-modules. (*Remark*: This means that localization is an exact functor from the category of R-modules to the category of $A^{-1}R$-modules)