MATH3195/M5195 EXERCISE SHEET 2

DUE: FEBRUARY 19, 2020

Problem 1. *Monomial orders:* Show that $<_{lex}$ is a monomial order on $K[x_1, \ldots, x_n]$ (or equivalently, on $\mathbb{N}^n$).

Problem 2. (a) Show that $\mathbb{R}[x,y]/(x^3 - y^2)$ is isomorphic to $\mathbb{R}[t^2, t^3]$. [Hint: First homomorphism theorem. First show that $f(x,y) = x^3 - y^2$ is in the kernel of the map φ as defined in the lecture. In order to see that $(f(x,y))$ is the full kernel, you may use the fact, that the kernel of φ is generated by elements of the form $x^a y^b - x^{a'} y^{b'}$, where $a, a', b, b' \in \mathbb{N}$. This fact can be proved using Gröbner bases methods]

(b) Is $(x^3 - y^2)$ a prime ideal in $\mathbb{R}[x,y]$? Explain!

Problem 3. (a) Show that the ideal $(x^4 - 5x^3 + 7x^2 - 5x + 6, x^4 + 2x^2 + 1, x^4 - 2x^3 + x^2 - 2x)$ in $\mathbb{R}[x]$ is maximal.

(b) Let R be a ring such that every element satisfies $x^n = x$ for some $n > 1$ (here the integer n depends on x). Show that every prime ideal in R is maximal.

Problem 4. (a) Consider $K[x,y,z]$ and order all monomials of degree less than or equal to 2 with respect to the following monomial orders: (i) $<_{lex}$, (ii) $<_{deglex}$, (iii) $<_{\lambda}$, where λ is a suitable linear form $\lambda : \mathbb{R}^3 \to \mathbb{R}$.

(b) Determine leading monomial and coefficient of the polynomial $f = x^4 + z^5 + x^3 z + yz^4 + x^2 y^2$ with respect to the momomial orders from (a).

Problem 5. Let R be a ring. Show that R is local if and only if the nonunits of R form a maximal ideal.

Problem 6. Let I be an ideal of R and A be a multiplicatively-closed subset of R. Show that:

(a) $A^{-1}I$ is an ideal of $A^{-1}R$;

(b) $\frac{x}{a} \in A^{-1}I$ if and only if there is some $b \in A$ with $xb \in I$;

(c) $A^{-1}I = A^{-1}R$ if and only if $I \cap A \neq \emptyset$;

(d) localisation commutes with quotients, that is

$$A^{-1}R/A^{-1}I \cong \overline{A}^{-1}(R/I),$$

where $\overline{A} = \{a + I : a \in A\}$.