Approximate Polynomial Greatest Common Divisor

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- numerically, an AGCD can be computed from Sylvester matrices:

Let

$$\hat{f}(x) = \sum_{i=0}^{m} \hat{a}_i x^{m-i} \;\; (\hat{a}_0 \hat{a}_m \neq 0) \quad \text{and} \quad \hat{g}(x) = \sum_{j=0}^{n} \hat{b}_j x^{n-j} \;\; (\hat{b}_0 \hat{b}_n \neq 0)$$

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$$u\hat{h} = \hat{f} \quad \text{and} \quad v\hat{h} = \hat{g}.$$

This equations can be written in the matrix-vector notation as

$$S_k(\hat{f}, \hat{g}) \begin{bmatrix} \mathbf{v} \\ -\mathbf{u} \end{bmatrix} = \mathbf{0}$$

where **u** denotes the vector of coefficients of u.

The matrix $S_k = S_k(\hat{f}, \hat{g})$ is the kth Sylvester subresultant matrix,

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The Sylvester matrix is the matrix $S = S_1 = \left[C_n(\hat{f}),\ C_m(\hat{g}) \right] \in \mathbb{R}^{(m+n)\times(m+n)}$.

The following statements hold:

i) $\text{rank}(S(\hat{f},\hat{g})) = m + n - k \iff \deg \hat{h} = k$,

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- TLS strategies by Winkler, Kaltofen, Zítko.

Inexact polynomials:

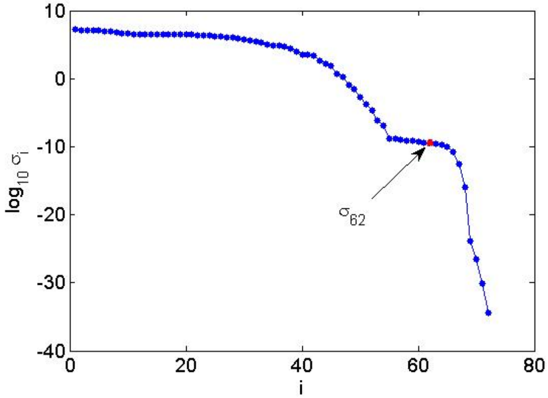
Assume that

$$\hat{f}(x) = \sum_{i=0}^{m} \hat{a}_i x^{m-i} \quad \text{and} \quad \hat{g}(x) = \sum_{j=0}^{n} \hat{b}_i x^{n-i}$$

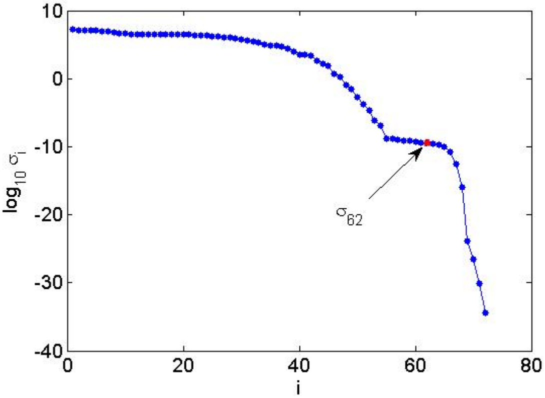
are exact polynomial. Componentwise perturbation, in particular

$$\begin{aligned} a_i &= \hat{a}_i + \epsilon \hat{a}_i c_i, \ i = 0, \ldots, m \\ b_i &= \hat{b}_i + \epsilon \hat{b}_i d_i, \ i = 0, \ldots, n, \end{aligned} \tag{1}$$

yields the inexact polynomials f and g, where c_i and d_i are random numbers in $[-1, 1]$ and $\text{SNR} = \frac{1}{\epsilon}$ is the signal-to-noise ratio.



Example of polynomials with the GCD of degree 10. The Sylvester matrix of these polynomials is $S \in \mathbb{R}^{72\times 72}$ and is of rank $r = 62$.



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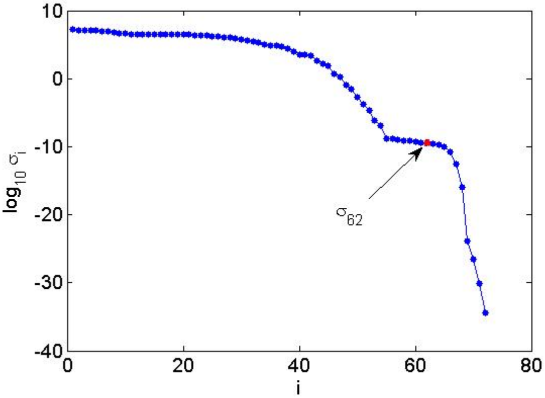

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Golub and Van Loan recommend to use $\theta = \varepsilon_{\text{mach}}\|S\| \approx 1.1 \times 10^{-9}$ that returns $r = 57$ and the numerical-rank gap criterion gives $r = 68$ that are incorrect results.

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where $\mathbf{c_k}$ is the first column of S_k and A_k is formed from the remaining $m + n - 2k + 1$ columns of S_k, $S_k = [\mathbf{c_k}, A_k]$.

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However, if the polynomials f and g are coprime, we can demand to compute the minimal corrections δf and δg of f and g so that $f + \delta f$ and $g + \delta g$ have a non-trivial GCD with the highest possible degree. Then, we set

$$\text{AGCD}(f, g) = \text{GCD}(f + \delta f, g + \delta g).$$

Let $\delta S_k = \delta S_k(\delta f, \delta g)$ be the Sylvester matrix of δf and δg, $\delta S_k = [\mathbf{h_k}, E_k]$, and let $\mathbf{z} = [\delta \mathbf{f}^T, \delta \mathbf{g}^T]^T$ be the vector of the coefficients of δf and δg.

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Then δf and δg are computed so that

$$(A_k + E_k)\mathbf{x} = \mathbf{c_k} + \mathbf{h_k}$$

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$$\min_{\mathbf{z},\mathbf{x}} \|\mathbf{z}\|$$

subject to $(A_k + E_k)\mathbf{x} = \mathbf{c_k} + \mathbf{h_k}$

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Methods:

- Structured Total Least Norm (STLN) of Park and Glick – for AGCD customized by Kaltofen; non-linear extension (SNTLN) is given by Winkler,
- Lagrange-Newton Structured TLS (LN-STLS) of Huffel.

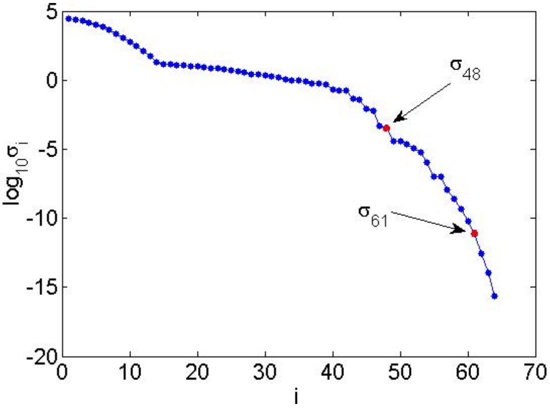
Exmaple. Consider the polynomials $\hat{f}$ and $\hat{g}$ of degree 32,

$$\hat{f}(x) = \prod_{i=1}^{8} \left[(x - r_1\alpha_i)^2 + r_1^2\beta_i^2 \right] \prod_{i=9}^{16} \left[(x - r_2\alpha_i)^2 + r_2^2\beta_i^2 \right],$$

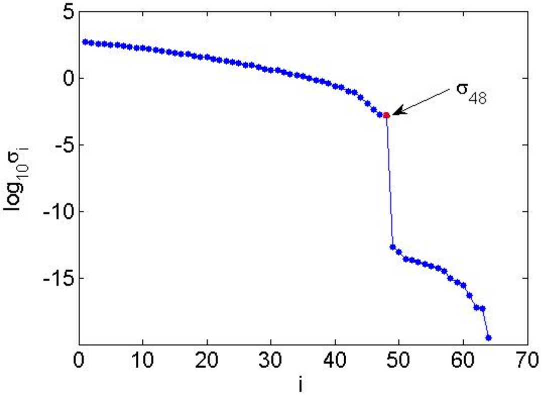
$$\hat{g}(x) = \prod_{i=1}^{16} \left[(x - r_1\alpha_i)^2 + r_1^2\beta_i^2 \right],$$

where $\alpha_i = \cos\left(\frac{\pi i}{m}\right)$, $\beta_i = \sin\left(\frac{\pi i}{m}\right)$, $i = 1, \dots, n$, $r_1 = 0.5$ and $r_2 = 1.5$. These polynomials have the exact GCD of degree 16. So the rank of the Sylvester matrix $S(f, g)$ is 48.

Suppose that a noise of the signal-to-noise ratio $\mathrm{SNR} = 10^8$ is componentwisely imposed to the coefficients of $\hat{f}$ and $\hat{g}$, thereby yielding the inexact polynomials f and g.



Singular values of the Sylvester matrix $S(f, g)$. For the threshold $\theta = \varepsilon_{mach}\|S\| \approx 10^{-12}$ the numerical rank is 61 that is incorrect. The correct numerical rank 48 can be revealed with $\theta \approx 10^{-4}$.



Singular values of $S(f,g) + \delta S(\delta f, \delta g)$ where δf and δg are obtained from the TLS problem solved by the LN-SLTS method.

Thesis review:

- sensitivity of the AGCD problem on noise,
- formulation of the AGCD problem as the structured TLS problem,
- overview of the methods for solving STLS (STLN and LN-STLS),
- modification of the Winkler's SNTLN method that is in a column pivoting in choosing a "better" TLS problem (a new contribution of the thesis),
- application of LN-STLS method on the AGCD problem (a new contribution of the thesis),
- developing a non-linear extension of the LN-STLS method (a new contribution of the thesis).

Thank you for you attention!

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