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7. Compact operators

a) Beispiele (d) in Werner's *Funktionalanalysis* on p. 70 in a bit modified form:

Let $k \in C([0,1]^2)$. For $f \in C([0,1])$ and $x \in [0,1]$ define

$$Tf(x) = \int_0^1 k(x,y)f(y)\,\mathrm{d}y.$$

- a1) Prove that for $f \in C([0,1])$ the function Tf is continuous.
- a2) Prove that the linear operator $T : C([0,1]) \to C([0,1])$ is bounded.
- a3) Prove that T is compact, i.e., $T : C([0,1]) \to C([0,1])$ maps bounded sets in $C([0,1])$ into relatively compact sets in $C([0,1])$. (Hint. Consider the set $\{f \in C([0,1]) : \|f\|_\infty \le 1\}$)

b) Aufgabe II.5.26 in Werner's *Funktionalanalysis* on p. 85: Let $k \in C([0,1]^2)$ and $T : C([0,1]) \to C([0,1])$ be defined by

$$Tf(x) = \int_0^x k(x,y)f(y)\,\mathrm{d}y.$$

Prove that T is compact.

c) Prove that compact operators are totally continuous: a compact operator K defined on a normed linear space X ($K : X \to X$) maps weakly convergent sequences to strongly convergent ones, i.e., if $x_n \rightharpoonup x$ weakly in X, then $Kx_n \to Kx$ strongly in X. *[Hint. For simplicity, assume that $x = \mathbf{0}$ is the null element of X and that Kx_n does not converge to $\mathbf{0}$ — try to find a contradiction. You may need the fact that if $x_n \rightharpoonup x$ weakly in X, then the sequence $\{\|x_n\|_X\}$ is bounded.]*

8. Find norm of an operator T, point spectrum $\sigma_p(T)$, spectrum $\sigma(T)$, and decide whether T is compact or not, where[1]

a) $T \in \mathcal{L}(l^\infty)$ is defined for $\{x_n\} \in l^\infty$ by

$$T : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, \ldots).$$

Solution. I think, that at this point you can easily show that $\|T\| = \|T\|_{\mathcal{L}(l^\infty)} = 1$. So, $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$. Let $|\lambda| \le 1$. Then, the equation $T(x_1, x_2, \ldots) = \lambda(x_1, x_2, \ldots)$ has a nontrivial solution $\{x_{n,\lambda}\} = (1, \lambda, \lambda^2, \lambda^3, \ldots)$. Is $\{x_{n,\lambda}\} \in l^\infty$? Jepp! So, for every λ satisfying $|\lambda| \le 1$, $T - \lambda I$ is not injection[2] and hence $\sigma_p(T) = \sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$. Moreover, $\sigma(T)$ is uncountable so there

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[1] Let X be a Banach space, $T : X \to X$, $T \in \mathcal{L}(X)$. Then $\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injection}\} = \{\lambda \in \mathbb{C} : \exists x \ne 0 \ s.t.\ Tx = \lambda x\}$; $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injection or it is not surjection (or both)}\}$. It holds that $\sigma_p(T) \subset \sigma(T)$ and $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \le \|T\|\}$ is compact subset of $\mathbb{C}$. If in addition T is compact, then $0 \in \sigma(T)$, $\sigma(T) = \{0\} \cup \sigma_p(T)$ and $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| \ge \varepsilon\}$ is finite for each $\varepsilon > 0$.

[2] T cannot be injection as it maps, for example, $(23, 1, 1, \ldots)$ and $(74, 1, 1, \ldots)$ to the same element $(1, 1, \ldots)$.

is no chance to have $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq \varepsilon\}$ finite for any small enough ε. Consequently, T is not compact.

b) $T \in \mathcal{L}(l^1)$ is defined for $\{x_n\} \in l^1$ by

$$T : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, \dots).$$

c) $T \in \mathcal{L}(l^2)$ is defined for $\{x_n\} \in l^2$ by

$$T : (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots).$$

d) $T \in \mathcal{L}(C([0,1]))$ is defined for $f \in C([0,1])$ by

$$Tf(x) = \int_0^x f(y)\,\mathrm{d}y, \quad x \in [0,1].$$

e) $T \in \mathcal{L}(L^2([0,1]))$ is defined for $f \in L^2([0,1])$ by

$$Tf(x) = x\int_0^1 f(y)\,\mathrm{d}y, \quad x \in [0,1].$$

f) $T \in \mathcal{L}(C([0,1]))$ is defined for $f \in C([0,1])$ by

$$T : f(x) \mapsto x^2 f(0), \quad x \in [0,1].$$

g*) $T \in \mathcal{L}(C([0,1]))$ is defined for $f \in C([0,1])$ by

$$T : f(x) \mapsto f(x^2), \quad x \in [0,1].$$

h) $T \in \mathcal{L}(l^2)$ is defined for $\{x_n\} \in l^2$ by

$$T\{x_n\} = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots).$$

i) $T \in \mathcal{L}(L^2([-1,1]))$ is defined for $f \in L^2([-1,1])$ by

$$T : f(x) \mapsto \int_{-1}^1 x^2 t f(t)\,\mathrm{d}t, \quad x \in [-1,1].$$

[Hint. In some examples, Arzelà-Ascoli Theorem can be useful.]

9. Other problems from the lecture

a) Assume that a sequence $\{a_n\}_n$ satisfies $0 \leq a_{n+m} \leq a_n a_m$ for all $n, m \in \mathbb{N}$. Then, $\{\sqrt[n]{a_n}\}_n$ converges to $a = \inf \sqrt[n]{a_n}$ (Lemma VI.1.4 in Werner's book).

b) Let H be a Hilbert space and $T \in \mathcal{L}(H)$ (normal). Then, $r(T) = \|T\|$. (Satz VI.1.7 in Werner's book).