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Some open problems from the lecture:

5. Lebesgue spaces

Let $\Omega \in \mathbb{R}^N$ such that $\mu(\Omega) < \infty$. Prove the following statements:

a) $L^\infty(\Omega) \subset L^q(\Omega) \subset L^p(\Omega) \subset L^1(\Omega)$ for any $1 < p < q < \infty$.

b) Let $f \in L^\infty(\Omega)$. Then $\|f\|_{L^\infty} = \lim_{p\to\infty}\|f\|_{L^p}$.

c) Let $1 \leq p_1 \leq p_2 \leq \ldots$ and suppose that $\lim_{k\to\infty} p_k = \infty$. Let $f \in \cap_{k=1}^\infty L^{p_k}(\Omega)$ and $\sup_{k\in\mathbb{N}}\|f\|_{L^{p_k}} < \infty$. Then $f \in L^\infty(\Omega)$.

d) In view of b) and c) show that any measurable function g on Ω such that

$$fg \in L^1(\Omega)$$

and

$$\left|\int_\Omega f(x)g(x)\,\mathrm{d}x\right| \leq M\|f\|_{L^1}$$

for arbitrary $f \in L^1(\Omega)$ and $M > 0$ satisfies that $g \in L^\infty(\Omega)$ and $\|g\|_{L^\infty} \leq M$.

6. Other problems

a) Contraction Mapping Principle (Banach Fixed Point Theorem) : Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point x in X (i.e., $T(x) = x$). *[Hint. Show that a sequence $\{x_n\}_{n=1}^\infty$ given by $x_{n+1} = T(x_n)$ starting with any $x_1 \in X$ is Cauchy sequence in (X, d) and that a limit of this sequence is the fixed point of T.]*

b) Let H be a Hilbert space and $K \subset H$ be a nonempty closed convex set. Show that the projection $P_K : H \to K$ (defined in the lecture) does not increase distance, i.e.,

$$|P_K(f_1) - P_K(f_2)| \leq |f_1 - f_2|, \quad \forall f_1, f_2 \in H.$$

c) Show that whenever $u \in C^1(\Omega)$ then the weak derivative coincides with the classical derivative.

d) Let H be a Hilbert space. Show that if $a(u, v)$ is a bilinear form with the property $a(v, v) \geq 0$ for all $v \in H$, then the function $v \mapsto a(v, v)$ is convex.

e) (Arzelà-Ascoli Theorem) Let (K, d) be a compact metric space and let $\mathcal{F} \subset C(K, \mathbb{R}^N)$. Then $\mathcal{F}$ is relatively compact (precompact) in $C(K, \mathbb{R}^N)$ if and only if $\mathcal{F}$ is bounded and uniformly equicontinuous, i.e.,

$$\forall \varepsilon > 0\ \exists \delta > 0\ \ \textit{such that}\ \ d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon\ \ \forall f \in \mathcal{F}. \tag{1}$$

f*) ("L^p version of Arzelà-Ascoli") Let $\mathcal{F} \subset L^p(\mathbb{R}^N)$ be a bounded set, $1 \le p < \infty$. Assume that

$$\forall \varepsilon > 0\ \exists \delta > 0\ \ such\ that\ \ \int_\Omega |f(x+h) - f(x)|^p < \varepsilon^p\ \ \forall f \in \mathcal{F}, \forall h \in \mathbb{R}^N,\ |h| < \delta$$

Then $\mathcal{F}|_\Omega$ is relatively compact (precompact) in $L^p(\Omega)$ for all measurable $\Omega \in \mathbb{R}^N$ with finite measure.[1]

Proof of Arzelà-Ascoli Theorem:

" $\Longrightarrow$ " Let $\mathcal{F}$ be a relatively compact set in $C(K, \mathbb{R}^N)$. By definition of the relative compactness, $\overline{\mathcal{F}}$ is compact, and so closed and bounded in $C(K, \mathbb{R}^N)$. Hence $\mathcal{F}$ is bounded in $C(K, \mathbb{R}^N)$.

Now, let $\varepsilon > 0$. Since $\overline{\mathcal{F}}$ is compact, then it is *precompact* (in some lit. *totally bounded*). By definition, there exists a *finite ε-net*[2] of $\overline{\mathcal{F}}$ in $C(K, \mathbb{R}^N)$; let's denote it by $I_\varepsilon = \{f_1, \dots, f_n\}$ for some $n < \infty$. Thus, for an arbitrarily chosen function $f \in \mathcal{F}$ we can find $i \in \{1, 2, \dots, n\}$ satisfying $\|f - f_i\|_\infty < \varepsilon$. On the other hand, since I_ε is finite, then it is equicontinuous.[3] So, let $x \in K$, then by (1) we can find $\delta > 0$ such that

$$\forall y \in K\ s.t.\ d(x,y) < \delta : |f_i(x) - f_i(y)| < \varepsilon \quad \forall i \in \{1, 2, \dots, n\}.$$

Hence, for $y \in K$ such that $d(x,y) < \delta$ we have

$$|f(x) - f(y)| \le |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < 3\varepsilon.$$

Since $f \in \mathcal{F}$ is arbitrary, then $\mathcal{F}$ is equicontinuous in $C(K, \mathbb{R}^N)$.

" $\Longleftarrow$ " Let $\mathcal{F} \subset C(K, \mathbb{R}^N)$ be bounded and uniformly equicontinuous and let $\{f_n\}$ be a sequence in $\mathcal{F}$.[4]

Let $n \in \mathbb{N}$ be fixed. Then we deduce from (1) that

$$\forall x \in K\ \exists \delta_x > 0\ \forall y \in K\ \forall f \in \mathcal{F},\ d(x,y) < \delta_x : |f(x) - f(y)| < \frac{1}{n}. \tag{2}$$

Since (K, d) is compact metric space, by definition, there exist a finite number of points $x_1^n, x_2^n, \dots, x_{k_n}^n \in K$ such that

$$K \subset \bigcup_{i=1}^{k_n} B(x_i^n, \delta_{x_i^n}) \tag{3}$$

where $B(x_i^n, \delta_{x_i^n})$ is a ball centred at x_i^n with radius $\delta_{x_i^n}$. Set

$$C = \{x_i^n : i \in \{1, 2, \dots, k_n\}, n \in \mathbb{N}\} =: \{c_1, c_2, \dots\}.$$

(we just rename the entries of C by c_1, c_2 and so on).

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[1] $\mathcal{F}|_\Omega$ denotes the restriction to Ω of the function in $\mathcal{F}$.

[2] We consider $X = C(K, \mathbb{R}^N)$ with the sup-norm. Let $\varepsilon > 0$ and $M \subset X$. A set E in X is called an ε-net of M if for every $u \in M$ there exists $u_\varepsilon \in E$ such that $\|u - u_\varepsilon\|_\infty < \varepsilon$.

[3] One can see (1) in the following way: $\sup_{f \in \mathcal{F}} |f(x_1) - f(x_2)| \to 0$ whenever $d(x_1, x_2) \to 0$ for $x_1, x_2 \in K$. By omitting $\sup_{f \in \mathcal{F}}$ in the limit we would end up with "just f being continuous function". Since the whole thing about *uniform equicontinuity* of $\mathcal{F}$ is in "$\sup_{f \in \mathcal{F}}$", if $\mathcal{F}$ is made of a finite number of continuous functions, then (1) is clearly satisfied.

[4] We would like to choose a Cauchy subsequence $\{f_{n_j}\}$ from $\{f_n\}$. Since $\{f_n\}$ is arbitrary, then, by definition, $\mathcal{F}$ is relatively compact in $C(K, \mathbb{R}^N)$.

(construction step) Since $\{f_n(c_1)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^N$, then by Bolzano-Weierstrass theorem[5] there exists an infinite set $N_1 \subset \mathbb{N}$ such that $\{f_n(c_1)\}_{n\in N_1}$ is Cauchy in $\mathbb{R}^N$. Now, let's assume that we have $N_l \subset \cdots \subset N_2 \subset N_1$ such that $\{f_n(c_i)\}_{n\in N_i}$ is Cauchy in $\mathbb{R}^N$ for each $i \in \{1,\ldots,l\}$. Since $\{f_n(c_{l+1})\}_{n\in N_l}$ is bounded, we can choose an infinite set $N_{l+1} \subset N_l$ such that $\{f_n(c_{l+1})\}_{n\in N_{l+1}}$ is Cauchy in $\mathbb{R}^N$. We find inductively the indices $n_1 < n_2 < \cdots$ such that $n_l \in N_l$ for $l \in \mathbb{N}$: first, choose an arbitrary $n_1 \in N_1$. If we have $n_1 < n_2 < \cdots < n_l$, then use the fact that N_{l+1} is infinite and find $n_{l+1} \in N_{l+1}$ such that $n_{l+1} > n_l$. *(end of the construction step)*

Note that for each $l \in \mathbb{N}$, the indices n_j for $j \geq l$ belong to N_l (since $N_j \subset N_l$ for $j \geq l$). Hence, $\{f_{n_j}(c_l)\}_{j=1}^{\infty}$ is selected from $\{f_n(c_l)\}_{n\in N_l}$ from the index n_l. Consequently, $\{f_{n_j}(c_l)\}_{j=1}^{\infty}$ is Cauchy in $\mathbb{R}^N$. In fact, $\{f_{n_j}\}_{j=1}^{\infty}$ is Cauchy in $C(K,\mathbb{R}^N)$. Indeed, let $\varepsilon > 0$ and find $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Since, by the above construction, the sequences

$$\{f_{n_j}(x_1^n)\}_{j=1}^{\infty}, \ldots, \{f_{n_j}(x_{k_n}^n)\}_{j=1}^{\infty}$$

are all Cauchy, there is an index $j_0 \in \mathbb{N}$ such that

$$\forall j, j' \in \mathbb{N},\ j, j' \geq j_0,\ \forall i \in \{1,\ldots,k_n\} : \left| f_{n_j}(x_i^n) - f_{n_{j'}}(x_i^n) \right| < \varepsilon. \tag{4}$$

Let $x \in K$. By (3), there is $i \in \{1,\ldots,k_n\}$ such that $x \in B(x_i^n, \delta_{x_i^n})$. Hence, for $j, j' \geq j_0$ we deduce from (2) and (4) that

$$\begin{aligned}\left| f_{n_j}(x) - f_{n_{j'}}(x) \right| &\leq \left| f_{n_j}(x) - f_{n_j}(x_i^n) \right| + \left| f_{n_j}(x_i^n) - f_{n_{j'}}(x_i^n) \right| + \left| f_{n_{j'}}(x_i^n) - f_{n_{j'}}(x) \right| \\ &\leq \frac{1}{n} + \varepsilon + \frac{1}{n} < 3\varepsilon.\end{aligned}$$

Hence, we proved that

$$\left\| f_{n_j} - f_{n_{j'}} \right\|_{\infty} = \sup_{x\in K} \left| f_{n_j}(x) - f_{n_{j'}}(x) \right| \leq 3\varepsilon$$

for $j, j' \geq j_0$, in other words, that $\{f_{n_j}\}$ is Cauchy in $C(K,\mathbb{R}^N)$. *Q.E.D.*

[5]BWT: every bounded sequence of real numbers (resp., in $\mathbb{R}^N$) has a convergent (and so Cauchy) subsequence.