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4. Approximation by continuous functions with compact support: Let $\Omega \subset \mathbb{R}^n$. Prove that $C_0(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$, i.e. for every $f \in L^p(\Omega)$ and $\varepsilon > 0$ there exists $g \in C_0(\Omega)$ such that $\|f - g\|_{L^p} < \varepsilon$.

a) Figure out why, without loss of generality, we may restrict ourself to real-valued and nonnegative functions $f \in L^p(\Omega)$.

b) For a given real-valued and nonnegative function $f \in L^p(\Omega)$ consider an approximating sequence of simple functions s_k (see below). Show that $s_k \in L^p(\Omega)$ for all k and that, for $\varepsilon > 0$ there exists $s \in \{s_k\}$ such that $\|f - s\|_{L^p} < \varepsilon/2$.

c) By using Luzin's theorem (see below) show that there exists $g \in C_0(\mathbb{R}^n)$ such that $\|s - g\|_{L^p} < \varepsilon/2$.

d) Conclude the statement.[1]

You may use the following theory:

➧ (Simple functions) A real valued function s on $\mathbb{R}^n$ is called *a simple function* if it is a linear combination of characteristic functions of sets in $\mathbb{R}^n$, i.e., if there are $\alpha_j \in \mathbb{R}$ and $A_j = \{x \in \mathbb{R}^n : s(x) = \alpha_j\}$ for $j = 1, \dots, m$ such that

$$s(x) = \sum_{j=1}^{m} \alpha_j \chi_{A_j};$$

s is measurable if and only if A_j, $j = 1, \dots, m$, are all measurable.

➧ (Approximation by simple functions[2]) Let f be a nonnegative, real-valued measurable function on $\Omega \subset \mathbb{R}^n$. There exists a sequence of nonnegative simple functions $\{s_k\}_{k=1}^{\infty}$ such that $s_k \nearrow f$ (monotonically increasing) pointwise on Ω.

➧ (Luzin's Theorem[3]) Let $\Omega \subset \mathbb{R}^n$ be measurable ($\mu(\Omega) < \infty$) and let f be measurable on Ω and $f = 0$ on $\mathbb{R}^n \backslash \Omega$. Let $\varepsilon > 0$. Then there exists a function $g \in C_0(\mathbb{R}^n)$ such that

$$\sup_{x \in \mathbb{R}^n} g(x) \le \sup_{x \in \mathbb{R}^n} f(x) \quad \text{and} \quad \mu(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}) < \varepsilon.$$

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[1] *Note that there exist other ways how to prove 4 (also without using Luzin's theorem). If you have another proof, please, feel free to present it.*

[2] Taken from Adams and Fournier's book *Sobolev spaces*, Thm 1.44 on p. 15.

[3] Taken from Adams and Fournier's book *Sobolev spaces*, Thm 1.42(f) on p. 15. For more complete results on the relation between measurability and continuity (including proofs) you can have a look into Lukeš and Malý's book *Measure and Integral*, Section 18 on pp. 75-76.