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1. The Fourier transform (on the real line)

Let the Fourier transform of an integrable function f be defined by

$$\widehat{f}(\xi) = \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x) \, \mathrm{d}x. \tag{1}$$

a) Find the Fourier transform of $f_1(x) = 1$, $f_2(x) = \chi_{[a,b]}(x)$, where $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$, $f_3(x) = e^{-|x|}$ and $f_4(x) = \sin(x)$. (1p)
[Hint. In the last case you may use Euler's formula to rewrite f_4.]

b) By applying the inverse Fourier transform

$$f(x) = \mathcal{F}^{-1}\{\widehat{f}(\xi)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \widehat{f}(\xi) \, \mathrm{d}\xi$$

show that for $x \in \mathbb{R}$

$$\int_0^{\infty} \frac{\cos(\xi x)}{1 + \xi^2} \, \mathrm{d}\xi = \frac{\pi}{2} e^{-|x|}. \qquad \text{(1p)}$$

c) By using the Fourier transform with respect to the spatial variable solve 1D heat equation $\varphi_t = \kappa \varphi_{xx}$ for $\varphi = \varphi(x, t)$, $x \in \mathbb{R}$ and $t \geq 0$ with the initial condition $\varphi(x, 0) = A\delta(x)$ where $\kappa > 0$, $A > 0$ and δ is the Delta function. (2p)
[Hint. Use/prove the fact that if $\widehat{f}(\xi) = \mathcal{F}\{f(x)\}$ then $\mathcal{F}\{f^n(x)\} = (i\xi)^n \widehat{f}(\xi)$ where f^n is the n-th derivative of f. Depending on your approach, you may also use $\int_{\mathbb{R}} e^{-x^2/2} \, \mathrm{d}x = \sqrt{2\pi}$.]

d) Prove that the Fourier transform $\widehat{f}$ of $f \in L^1(\mathbb{R})$ is a uniformly continuous function. (1p)

2. Norms of linear operators

Let $\mathcal{X}$ and $\mathcal{Y}$ be two normed linear spaces (NLS). Find norms of the following linear functionals/operators $T : \mathcal{X} \to \mathcal{Y}$ (each ex. for 1p)[1]; if exists, find $g \in \mathcal{X}$ such that $\|T(g)\| = \|T\|$ or prove that such g does not exist.

a) Fourier transform $T = \mathcal{F}$ in (1) on $L^1(\mathbb{R})$;

b) functional T defined on $C([0, 1])$ (with max-norm) by

$$T : f \mapsto \int_0^1 f(\sqrt{t}) \, \mathrm{d}t;$$

c) functional T defined on $C([-1, 1])$ by

$$T : f \mapsto \int_{-1}^{0} f(t) \, \mathrm{d}t - \int_0^1 f(t) \, \mathrm{d}t;$$

[1] Notation: we say that $T \in \mathcal{L}(\mathcal{X}, \mathbb{K})$ is a functional if $\mathcal{Y} = \mathbb{K} := \mathbb{R}$ (or $\mathbb{C}$), then $\|T\| := \sup\{|T(\varphi)| : \|\varphi\| \leq 1\}$ (more precisely, $\|T\|_{\mathcal{L}(\mathcal{X},\mathbb{K})} := \sup\{|T(\varphi)| : \|\varphi\|_{\mathcal{X}} \leq 1\}$); on the other hand $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is an operator if $\mathcal{Y}$ is a NLS other than $\mathbb{K}$, then $\|T\| := \sup\{\|T(\varphi)\| : \|\varphi\| \leq 1\}$ (more precisely, $\|T\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} := \sup\{\|T(\varphi)\|_{\mathcal{Y}} : \|\varphi\|_{\mathcal{X}} \leq 1\}$).

d) functional T defined on the space $\mathcal{C} = \{f \in C([0,1]) : f(0) = 0\}$ by

$$T : f \mapsto \int_0^1 f(t)\,\mathrm{d}t;$$

e) functional T defined on $l^2 = \{\{x_n\}_{n=1}^{\infty} : \|\{x_n\}_n\| = \left(\sum_1^{\infty} |x_n|^2\right)^{1/2} < \infty\}$ by

$$T : \{x_n\}_n \mapsto x_1 + x_2;$$

f) functional T defined on l^2 by

$$T : \{x_n\}_n \mapsto \sum_{n=1}^{\infty} \frac{x_n}{n}.$$

3. Weak derivatives

Decide if the following functions $f : \mathbb{R} \to \mathbb{R}$ are weakly differentiable (in $L^1_{\mathrm{loc}}(\mathbb{R})$); if f is weakly differentiable, find a weak derivative of f:

a)

$$f : x \mapsto |x| \quad \forall x \in \mathbb{R}. \qquad \text{(1p)}$$

b)

$$f : x \mapsto \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \qquad \text{(1p)}$$

c)

$$f : x \mapsto \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases} \qquad \text{(1p)}$$

d) the Cantor function: $f(x) = 0$ if $x \leq 0$; $f(x) = 1$ if $x \geq 1$; and $f(x) = 1/2$ for $1/3 \leq x \leq 2/3$, $f(x) = 1/4$ for $1/9 \leq x \leq 2/9$, $f(x) = 3/4$ for $7/9 \leq x \leq 8/9$, ... (1p)
[Animation on wiki: `https://upload.wikimedia.org/wikipedia/commons/7/7f/Cantor_function.gif`*]*

e) $f : (-1, 2) \to \mathbb{R}$ such that

$$f : x \mapsto \begin{cases} |x| & \text{if } x < 1, \\ 1 - x^2 & \text{if } x \geq 1. \end{cases} \qquad \text{(1p)}$$