# Convex regularization of hybrid discrete-continuous inverse problems

**Christian Clason**[1] Karl Kunisch[2]

[1]Faculty of Mathematics, Universität Duisburg-Essen
[2]Institute of Mathematics and Scientific Computing, Karl-Franzens-Universität Graz

Workshop Recent Developments in Inverse Problems
WIAS, Berlin, September 18, 2015

# Motivation: discrete optimization

## $L^0$ penalty

$$\|u\|_0 := \int_\Omega |u(x)|_0 \, dx \qquad |t|_0 := \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t \neq 0 \end{cases}$$

- Lebesgue measure of support of $u$
- popular in sparse optimization
- binary penalty $\rightsquigarrow$ combinatorial optimization
- difficulty: non-smooth, non-convex, not lower-semicontinuous
- not a norm $\rightsquigarrow$ not coercive

# Binary penalties

$$\min_u \mathcal{F}(u) + \mathcal{G}(u)$$

- $\mathcal{F}(u)$ tracking or discrepancy term

1 $\mathcal{G}(u)$ sparsity penalty [Ito, Kunisch 2012]

$$\mathcal{G}(u) = \frac{\alpha}{2}\|u\|_{L^2}^2 + \beta\|u\|_0$$

  - $\rightsquigarrow u(x) = 0$ almost everywhere
  - separate penalization of support ($\beta$), magnitude ($\alpha$)
  - $\rightsquigarrow \alpha > 0$ necessary

# Binary penalties

$$\min_u \mathcal{F}(u) + \mathcal{G}(u)$$

2 $\mathcal{G}(u)$ multi-bang penalty [Clason, Kunisch 2013]

$$\mathcal{G}(u) = \int_\Omega \frac{\alpha}{2}|u(x)|^2 + \beta \prod_{i=1}^{d} |u(x) - u_i|_0 \, dx$$

- $\rightsquigarrow u(x) \in \{u_1, \ldots, u_d\}$ almost everywhere
- motivation: discrete control (voltages, velocities, materials)
- $\beta > 0$ penalizes *free arc* where $u(x) \neq u_i$
- $\alpha > 0$ penalizes magnitude of $u(x) = u_i$

# Binary penalties

UNIVERSITÄT DUISBURG ESSEN
Open-Minded

$$\min_u \mathcal{F}(u) + \mathcal{G}(u)$$

3 $\mathcal{G}(u)$ switching penalty, $u = (u_1, u_2)$ [Clason, Ito, Kunisch 2014]

$$\mathcal{G}(u) = \int_0^T \frac{\alpha}{2} |u(t)|_2^2 + \beta |u_1(t) u_2(t)|_0 \, dt$$

- $\rightsquigarrow u_1(t) u_2(t) = 0$ almost everywhere
- $\beta > 0$ penalizes free arc where $u_1 \neq 0$ *and* $u_2 \neq 0$
- $\alpha > 0$ penalizes magnitude of active $u_i$

# Convex relaxation: motivation

$f : \mathbb{R} \to \mathbb{R}$ differentiable:

- derivative:

$$f'(u) = \lim_{h \to 0} \frac{f(u+h) - f(u)}{h}$$

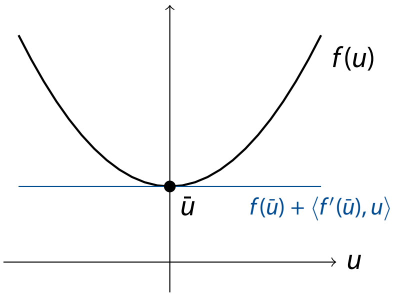
- geometrically:

  $f'(u)$ tangent slope

- Fermat principle:

  $f(\bar{u}) = \min_u f(u) \Rightarrow f'(\bar{u}) = 0$

- calculus for $f'$



# Convex relaxation: motivation

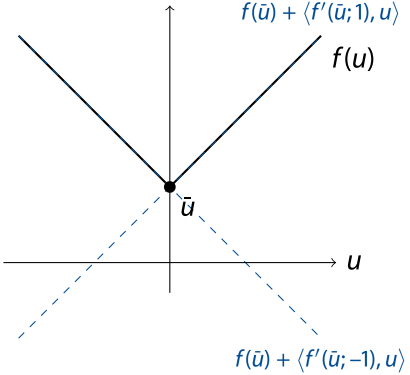
$f : \mathbb{R} \to \mathbb{R}$ not differentiable, convex:

- directional derivative:

$$f'(u;h) = \lim_{t\to 0^+} \frac{f(u+th) - f(u)}{t}$$

- but: for all $h$,

$$f'(\bar{u};h) \neq 0$$



# Convex relaxation: motivation

$f : \mathbb{R} \to \mathbb{R}$ not differentiable, convex:

- subdifferential:

  $$\partial f(u) = \left\{ u^* : \langle u^*, h \rangle \leqslant f'(u; h) \right\}$$

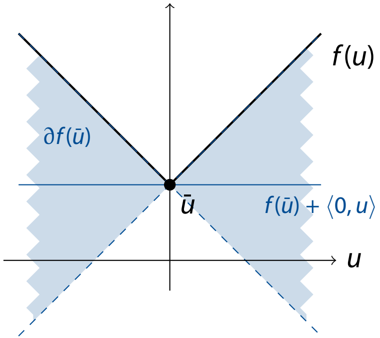
- geometrically:

  $\partial f(u)$ set of tangent slopes

- Fermat principle:

  $f(\bar{u}) = \min_u f(u) \Rightarrow 0 \in \partial f(\bar{u})$

- calculus for $\partial f$



# Fenchel duality



$f : V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ convex, $V$ Banach space, $V^*$ dual space

- subdifferential

$$\partial f(\bar{v}) = \left\{ v^* \in V^* : \langle v^*, v - \bar{v} \rangle_{V^*,V} \leqslant f(v) - f(\bar{v}) \quad \text{for all } v \in V \right\}$$

- Fenchel conjugate (always convex)

$$f^* : V^* \to \overline{\mathbb{R}}, \qquad f^*(v^*) = \sup_{v \in V} \langle v^*, v \rangle_{V^*,V} - f(v)$$

- "convex inverse function theorem":

$$v^* \in \partial f(v) \quad \Leftrightarrow \quad v \in \partial f^*(v^*)$$

# Fenchel duality: application



$$\mathcal{F}(\bar{u}) + \mathcal{G}(\bar{u}) = \min_u \mathcal{F}(u) + \mathcal{G}(u)$$

1. Fermat principle: $0 \in \partial\left(\mathcal{F}(\bar{u}) + \mathcal{G}(\bar{u})\right)$
2. sum rule: $0 \in \partial\mathcal{F}(\bar{u}) + \partial\mathcal{G}(\bar{u})$, i.e., there is $\bar{p} \in V^*$ with

$$\begin{cases} -\bar{p} \in \partial\mathcal{F}(\bar{u}) \\ \bar{p} \in \partial\mathcal{G}(\bar{u}) \end{cases}$$

3. Fenchel duality:

$$\begin{cases} -\bar{p} \in \partial\mathcal{F}(\bar{u}) \\ \bar{u} \in \partial\mathcal{G}^*(\bar{p}) \end{cases}$$

# Fenchel duality: example



- $\mathcal{G} : V \to \mathbb{R}, \quad v \mapsto \|v\|_V$:

$$\mathcal{G}^* : V^* \to \overline{\mathbb{R}}, \quad v^* \mapsto \delta_{\{\|\cdot\|_{V^*} \leqslant 1\}}(v^*) := \begin{cases} 0 & \text{if } \|v^*\|_{V^*} \leqslant 1 \\ \infty & \text{else} \end{cases}$$

- $\mathcal{G} : V \to \overline{\mathbb{R}}, \quad v \mapsto \delta_{\{\|\cdot\|_V \leqslant 1\}}(v)$:

$$\partial \mathcal{G}(\bar{v}) = \left\{ v^* \in V^* : \langle v^*, v - \bar{v} \rangle_{V^*,V} \leqslant 0 \quad \text{for all} \quad \|v\|_V \leqslant 1 \right\}$$

$\rightsquigarrow$ norm minimization equivalent to smooth optimization with constraints

# Convex relaxation

Consider $\mathcal{F}$ convex, $\mathcal{G}$ non-convex

$$J(\bar{u}) := \mathcal{F}(\bar{u}) + \mathcal{G}(\bar{u}) = \min_u \mathcal{F}(u) + \mathcal{G}(u)$$

Optimality(?) conditions:

$$\begin{cases} -\bar{p} \in \partial\mathcal{F}(\bar{u}) \\ \bar{u} \in \partial\mathcal{G}^*(\bar{p}) \end{cases}$$

- Fenchel conjugate always convex, lower semi-continuous
- $\rightsquigarrow$ well-defined, solution $\bar{u}$ exists (minimizes $\mathcal{F}(u) + \mathcal{G}^{**}(u)$)
- but: $\bar{u}$ in general not minimizer of $J \rightsquigarrow$ sub-optimal

# Smoothing

UNIVERSITÄT DUISBURG ESSEN

Open-Minded

$\mathcal{G}$ non-convex $\rightsquigarrow$ subdifferential $\partial\mathcal{G}^*$ set-valued $\rightsquigarrow$ local smoothing

assume $u, p \in L^2$ Hilbert space: consider for $\gamma > 0$

**Proximal mapping**

$$\operatorname{prox}_{\gamma\mathcal{G}^*}(p) = \arg\min_w \mathcal{G}^*(w) + \frac{1}{2\gamma}\|w - p\|^2$$

- single-valued, Lipschitz continuous
- coincides with resolvent $(\mathrm{Id} + \gamma\partial\mathcal{G}^*)^{-1}(p)$

# Smoothing

### Proximal mapping

$$\text{prox}_{\gamma\mathcal{G}^*}(p) = \arg\min_w \mathcal{G}^*(w) + \frac{1}{2\gamma}\|w - p\|^2$$

### Complementarity formulation of $u \in \partial\mathcal{G}^*(p)$

$$u = \frac{1}{\gamma}\left((p + \gamma u) - \text{prox}_{\gamma\mathcal{G}^*}(p + \gamma u)\right)$$

- equivalent for every $\gamma > 0$
- single-valued, Lipschitz continuous, implicit

# Smoothing

UNIVERSITÄT DUISBURG ESSEN

Open-Minded

**Proximal mapping**

$$\mathrm{prox}_{\gamma\mathcal{G}^*}(p) = \arg\min_w \mathcal{G}^*(w) + \frac{1}{2\gamma}\|w-p\|^2$$

**Moreau–Yosida regularization of $u \in \partial\mathcal{G}^*(p)$**

$$u = \frac{1}{\gamma}\left(p - \mathrm{prox}_{\gamma\mathcal{G}^*}(p)\right) =: \partial\mathcal{G}^*_\gamma(p)$$

- not equivalent, but $\partial\mathcal{G}^*_\gamma(p) \to \partial\mathcal{G}^*(p)$ as $\gamma \to 0$
- single-valued, Lipschitz continuous, explicit
  $\rightsquigarrow$ nonsmooth operator equation, Newton method

# Smoothing: example

UNIVERSITÄT DUISBURG ESSEN

Open-Minded

$\mathcal{G}^* : V^* \to \overline{\mathbb{R}}, \quad p \mapsto \delta_{\{\|\cdot\|_{V^*} \leqslant 1\}}(p)$:

- Proximal mapping:

$$\operatorname{prox}_{\gamma\mathcal{G}^*}(p) = \operatorname{proj}_{\{\|\cdot\|_{V^*} \leqslant 1\}}(p)$$

- Complementarity formulation: soft-shrinkage

- Moreau–Yosida regularization ($V^* = L^\infty(\Omega)$):

$$\partial\mathcal{G}^*_\gamma(p) = \frac{1}{\gamma}\Big(\max(0, p-1)) + \min(0, p+1)\Big)$$

(max, min pointwise almost everywhere)

# Generalized Newton method

Consider Banach spaces $X, Y$, mapping $F : X \to Y$

**Newton method for $F(x) = 0$**

- choose $x^0 \in X$ (close to solution $x^*$)
- for $k = 0, 1, \dots$
  1. choose $M_k \in \mathcal{L}(X, Y)$ invertible
  2. solve for $s^k$:
  $$M_k s^k = -F(x^k)$$
  3. set $x^{k+1} = x^k + s^k$

# Generalized Newton method

### Newton method for $F(x) = 0$

- choose $x^0 \in X$ (close to solution $x^*$)
- for $k = 0, 1, \ldots$
  1. choose $M_k \in \mathcal{L}(X, Y)$ invertible
  2. solve for $s^k$:
  $$M_k s^k = -F(x^k)$$
  3. set $x^{k+1} = x^k + s^k$

- convergence, i.e., $\|x^k - x^*\|_X \to 0$?
- superlinear convergence, i.e., $\frac{\|x^{k+1} - x^*\|_X}{\|x^k - x^*\|_X} \to 0$?

# Convergence of Newton method



Set $d^k = x^k - x^* \rightsquigarrow$

$$\|x^{k+1} - x^*\|_X = \|M_k^{-1}(F(x^* + d^k) - F(x^*) - M_k d^k)\|_X$$

$\rightsquigarrow$ superlinear convergence if

1. regularity condition

$$\|M_k^{-1}\|_{\mathcal{L}(Y,X)} \leqslant C \qquad \text{for all } k$$

2. approximation condition

$$\lim_{\|d^k\|_X \to 0} \frac{\|F(x^* + d^k) - F(x^*) - M_k d^k\|_Y}{\|d^k\|_X} = 0$$

# Semismooth Newton method

UNIVERSITÄT DUISBURG ESSEN
Open-Minded

**Goal:** define Newton derivative $M_k = D_N F$ such that

$$x^{k+1} = x^k - D_N F(x^k)^{-1} F(x^k)$$

converges superlinearly for $F(x) = 0$ nonsmooth

- $\mathbb{R}^n$: $F$ Lipschitz $\rightsquigarrow$ $D_N F$ from Clarke subdifferential
  [Mifflin 1977, Kummer 1992, Qi/Sun 1993]
- function space: no Clarke subdifferential
  $\rightsquigarrow$ define $D_N F$ via approximation condition
  [Chen/Nashed/Qi 2000, Hintermüller/Ito/Kunisch 2002]
- $f : \mathbb{R}^N \to \mathbb{R}$ semismooth $\rightsquigarrow$ superposition operator
  $F : L^p(\Omega) \to L^q(\Omega)$ semismooth iff $p > q$
  [Ulbrich 2002/03/11, Schiela 2008]

# Semismooth functions: example



- $f : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \max(0, t)$

$$D_N f(t) \in \partial_C f(t) = \begin{cases} \{0\} & t < 0 \\ \{1\} & t > 0 \\ [0,1] & t = 0 \end{cases}$$

- $F : L^p(\Omega) \to L^q(\Omega), \quad u(x) \mapsto \max(0, u(x)), \quad p > q$

$$[D_N F(u)h](x) = \begin{cases} 0 & u(x) < 0 \\ h(x) & u(x) \geqslant 0 \end{cases}$$

$\rightsquigarrow$ Moreau–Yosida regularization semismooth

# Numerical solution: summary

For nonconvex $\mathcal{G} : L^2(\Omega) \to \mathbb{R}, \quad \mathcal{G}(u) = \int_\Omega g(u(x))\,dx,$

1. compute Fenchel conjugate $g^*(q)$
2. compute subdifferential $\partial g^*(q)$
3. compute proximal mapping $\operatorname{prox}_{\gamma g^*}(q)$
4. compute Moreau–Yosida regularization $\partial g^*_\gamma(q)$,
   Newton derivative $D_N \partial g^*_\gamma(q)$
5. $\rightsquigarrow$ semismooth Newton method, continuation in $\gamma$ for
   superposition operator $\partial \mathcal{G}^*_\gamma(p)(x) = \partial g^*_\gamma(p(x))$

# Formulation

UNIVERSITÄT DUISBURG ESSEN

Open-Minded

$$\begin{cases} \min\limits_{u\in L^2(\Omega)} \dfrac{1}{2}\|y-y^\delta\|_{L^2}^2 + \dfrac{\alpha}{2}\|u\|_{L^2}^2 + \beta\displaystyle\int_\Omega \prod_{i=1}^{d} |u(x)-u_i|_0\,dx \\ \text{s.t. } Ay = u, \qquad u_1 \leqslant u(x) \leqslant u_d \end{cases}$$

- $u_1 < \cdots < u_d$ given parameter values $(d > 2)$
- $y^\delta \in L^2(\Omega)$ noisy data
- $A : V \to V^*$ isomorphism for Hilbert space $V \hookrightarrow L^2(\Omega) \hookrightarrow V^*$

  (e.g., elliptic differential operator with boundary conditions)
- $\rightsquigarrow \mathcal{F}(u) = \frac{1}{2}\|A^{-1}u - y^\delta\|_{L^2}^2$ convex, Fréchet-differentiable

# Fenchel conjugate

UNIVERSITÄT DUISBURG ESSEN

Open-Minded

$$g : \mathbb{R} \to \overline{\mathbb{R}}, \qquad v \mapsto \frac{\alpha}{2} v^2 + \beta \prod_{i=1}^{d} |v - u_i|_0 + \delta_{[u_1, u_d]}(v)$$

$$g^* : \mathbb{R} \to \overline{\mathbb{R}}, \qquad q \mapsto \sup_v \; q\,v - g(v)$$

Case differentiation: sup attained at $\bar{v}$,

$$g^*(q) = \begin{cases} qu_i - \frac{\alpha}{2} u_i^2 & \bar{v} = u_i, \quad 1 \leqslant i \leqslant d \\ \frac{1}{2\alpha} q^2 - \beta & \bar{v} \neq u_i, \quad 1 \leqslant i \leqslant d \end{cases}$$

# Fenchel conjugate

$$g^*(q) = \begin{cases} qu_i - \frac{\alpha}{2}u_i^2 & q \in \overline{Q}_i, \quad 1 \leqslant i \leqslant d \\ \frac{1}{2\alpha}q^2 - \beta & q \in \overline{Q}_0 \end{cases}$$

$$Q_1 := \left\{ q : q - \alpha u_1 < \sqrt{2\alpha\beta} \,\wedge\, q < \frac{\alpha}{2}(u_1 + u_2) \right\}$$

$$Q_i := \left\{ q : |q - \alpha u_i| < \sqrt{2\alpha\beta} \,\wedge\, \frac{\alpha}{2}(u_{i-1} + u_i) < q < \frac{\alpha}{2}(u_i + u_{i+1}) \right\}$$

$$Q_d := \left\{ q : q - \alpha u_d > \sqrt{2\alpha\beta} \,\wedge\, \frac{\alpha}{2}(u_d + u_{d-1}) < q \right\}$$

$$Q_0 := \left\{ q : |q - \alpha u_j| > \sqrt{2\alpha\beta} \text{ for all } j \,\wedge\, \alpha u_1 < q < \alpha u_d \right\}$$

# Fenched conjugate

$$g^*(q) = \begin{cases} qu_i - \frac{\alpha}{2}u_i^2 & q \in \overline{Q}_i, \quad 1 \leqslant i \leqslant d \\ \frac{1}{2\alpha}q^2 - \beta & q \in \overline{Q}_0 \end{cases}$$

continuous, piecewise differentiable:

$$\partial g^*(q) = \begin{cases} \{u_i\} & q \in Q_i,\ 1 \leqslant i < d \\ \left\{\frac{1}{\alpha}q\right\} & q \in Q_0 \\ [u_i, u_{i+1}] & q \in \overline{Q}_i \cap \overline{Q}_{i+1},\ 1 \leqslant i < d \\ \left[\min\left\{u_i, \frac{1}{\alpha}q\right\}, \max\left\{u_i, \frac{1}{\alpha}q\right\}\right] & q \in \overline{Q}_i \cap \overline{Q}_0,\ 1 \leqslant i \leqslant d \end{cases}$$

(no explicit dependence on $\beta$!)

# Optimality system

UNIVERSITÄT DUISBURG ESSEN
Open-Minded

$$\bar{p} = S^*(y^\delta - S\bar{u})$$
$$\bar{u} \in \partial\mathcal{G}^*(\bar{p})$$
$$= \begin{cases} \{u_i\} & \bar{p}(x) \in Q_i \\ \left\{\frac{1}{\alpha}\bar{p}(x)\right\} & \bar{p}(x) \in Q_0 \\ [u_i, u_{i+1}] & \bar{p}(x) \in \overline{Q}_i \cap \overline{Q}_{i+1} \\ \left[\min\left\{u_i, \frac{1}{\alpha}\bar{p}(x)\right\}, \max\left\{u_i, \frac{1}{\alpha}\bar{p}(x)\right\}\right] & \bar{p}(x) \in \overline{Q}_i \cap \overline{Q}_0 \end{cases}$$

- $S: u \mapsto y$ parameter-to-observation mapping, $S^*$ adjoint
- necessary conditions for $\min_u \mathcal{F}(u) + \mathcal{G}^{**}(u)$ (convex, l.s.c.)
- $\rightsquigarrow$ unique solution $(\bar{u}, \bar{p}) \in L^2(\Omega) \times L^2(\Omega)$

# Structure of solution

$$\Omega = \mathcal{A} \cup \mathcal{F} \cup \mathcal{S}$$

- multi-bang arc $\mathcal{A} = \bigcup_{i=1}^{d} \{x : \bar{u}(x) = u_i\}$
- free arc $\mathcal{F} = \left\{x : \bar{u}(x) = \frac{1}{\alpha}\bar{p}(x) \neq u_i\right\}$
- singular arc $\mathcal{S} = \left\{x : \bar{u}(x) \notin \left\{u_i, \frac{1}{\alpha}\bar{p}(x)\right\}\right\}$

# Generalized multi-bang principle

- if $\beta$ sufficiently large: $Q_0 = \emptyset$, free arc

$$\mathcal{F} \subset \{\bar{p}(x) \in Q_0\} = \emptyset$$

- singular arc corresponds to set-valued subdifferential:

$$\mathcal{S} = \left\{ \bar{p}(x) \in \bigcup_{i=1}^{d-1} (\overline{Q}_i \cap \overline{Q}_{i+1}) \cup \bigcup_{i=1}^{d} (\overline{Q}_i \cap \overline{Q}_0) \right\}$$
$$\subset \left\{ \bar{p}(x) \in \left\{ \tfrac{\alpha}{2}(u_i + u_{i+1}), \alpha u_i - \sqrt{2\alpha\beta}, \alpha u_i + \sqrt{2\alpha\beta} \right\} \right\}$$

- for suitable $A$, $\bar{p}(x)$ constant implies $[A^*\bar{p}](x) = [y^\delta - \bar{y}](x) = 0$

$\rightsquigarrow$ $|\{x : \bar{y}(x) = y^\delta(x)\}| = 0 \;\Rightarrow\; \bar{u} \in \{u_1, \ldots u_d\}$ a. e., true multi-bang

# (Sub)optimality



- duality gap for non-convex $\mathcal{G}$:

$$\mathcal{G}(\bar{u}) + \mathcal{G}^*(\bar{p}) - \langle \bar{p}, \bar{u} \rangle \leqslant \beta|\mathcal{S}|$$

(pointwise gap of $\beta$ where $\partial g^*(\bar{p}(x))$ set-valued)

- $\rightsquigarrow$ in general: $\bar{u}$ sub-optimal:

$$J(\bar{u}) \leqslant J(u) + \beta|\mathcal{S}| \qquad \text{for all } u$$

- but: $\bar{u}$ true multi-bang $\rightsquigarrow |\mathcal{S}| = 0 \rightsquigarrow \bar{u}$ optimal

# Moreau–Yosida regularization

$$\partial g_\gamma^*(q) = \begin{cases} u_i & q \in Q_i^\gamma \\ \frac{1}{\alpha+\gamma} q & q \in Q_0^\gamma \\ \frac{1}{\gamma}\left(q - (\alpha u_i + \sqrt{2\alpha\beta})\right) & q \in Q_{i0}^\gamma \\ \frac{1}{\gamma}\left(q - \frac{\alpha}{2}(u_i + u_{i+1})\right) & q \in Q_{i,i+1}^\gamma \end{cases}$$

$$Q_i^\gamma = \Big\{ q : |q - (\alpha+\gamma)u_i| < \sqrt{2\alpha\beta} \wedge \frac{\alpha}{2}\left(u_{i-1} + \left(1 + \frac{2\gamma}{\alpha}\right) u_i\right) < q < \frac{\alpha}{2}\left(\left(1 + \frac{2\gamma}{\alpha}\right) u_i + u_{i+1}\right) \Big\}$$

$$Q_0^\gamma = \left\{ q : |q - (\alpha+\gamma)u_j| > \sqrt{2\alpha\beta} \wedge (\alpha+\gamma)u_1 < q < (\alpha+\gamma)u_d \right\}$$

$$Q_{i,i+1}^\gamma = \left\{ q : \frac{\alpha}{2}\left(\left(1 + \frac{2\gamma}{\alpha}\right) u_i + u_{i+1}\right) \leqslant q \leqslant \frac{\alpha}{2}\left(u_i + \left(1 + \frac{2\gamma}{\alpha}\right) u_{i+1}\right) \right\}$$

$$Q_{i0}^\gamma = \left\{ q : \sqrt{2\alpha\beta} \leqslant q - (\alpha+\gamma)u_i \leqslant \left(1 + \frac{\gamma}{\alpha}\right) \sqrt{2\alpha\beta} \right\}$$

# Regularized optimality system

$$\begin{cases} p_\gamma = S^*(y^\delta - Su_\gamma) \\ u_\gamma = \partial \mathcal{G}_\gamma^*(p_\gamma) \end{cases}$$

- $\partial \mathcal{G}_\gamma^*$ maximal monotone $\rightsquigarrow$ unique solution $(u_\gamma, p_\gamma)$
- $(u_\gamma, p_\gamma) \rightharpoonup (\bar{u}, \bar{p})$ as $\gamma \to 0$
- $\partial g_\gamma^*$ Lipschitz continuous, piecewise $C^1$, norm gap $V \hookrightarrow L^2(\Omega)$
- $\rightsquigarrow$ semismooth Newton method

# Regularized optimality system

UNIVERSITÄT DUISBURG ESSEN

Open-Minded

$$\begin{cases} A^* p_\gamma = y^\delta - y_\gamma \\ A y_\gamma = \mathcal{G}_\gamma^*(p_\gamma) \end{cases}$$

- $\partial \mathcal{G}_\gamma^*$ maximal monotone $\rightsquigarrow$ unique solution $(u_\gamma, p_\gamma)$
- $(u_\gamma, p_\gamma) \rightharpoonup (\bar{u}, \bar{p})$ as $\gamma \to 0$
- $\partial g_\gamma^*$ Lipschitz continuous, piecewise $C^1$, norm gap $V \hookrightarrow L^2(\Omega)$
- $\rightsquigarrow$ semismooth Newton method
- introduce $y_\gamma = S u_\gamma$, eliminate $u_\gamma = \mathcal{G}_\gamma^*(p_\gamma)$

# Semismooth Newton method

$$\begin{pmatrix} \mathrm{Id} & A^* \\ A & D_N \mathcal{G}_\gamma^*(p) \end{pmatrix} \begin{pmatrix} \delta p \\ \delta y \end{pmatrix} = - \begin{pmatrix} A^* p + y - y^\delta \\ Ay - \mathcal{G}_\gamma^*(p) \end{pmatrix}$$

$$[D_N \mathcal{G}_\gamma^*(p)\delta p](x) = \begin{cases} \frac{1}{\alpha+\gamma}\delta p(x) & p(x) \in Q_0^\gamma \\ \frac{1}{\gamma}\delta p(x) & p(x) \in Q_{i,i+1}^\gamma \\ 0 & \text{else} \end{cases}$$

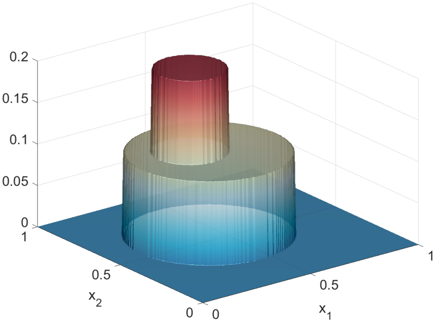
- symmetric, but: local convergence
- $\rightsquigarrow$ continuation in $\gamma \to 0$
- $\rightsquigarrow$ backtracking line search based on residual norm
- only number of sets $Q_i$ depends on $d$ $\rightsquigarrow$ linear complexity

# Example: linear inverse problem

UNIVERSITÄT DUISBURG ESSEN
Open-Minded

- $\Omega = [0,1]^2, \quad A = -\Delta$
- $$u^\dagger(x) = u_1 + u_2\chi_{\{x:(x_1-0.45)^2+(x_2-0.55)^2<0.1\}}(x) + (u_3-u_2)\chi_{\{x:(x_1-0.4)^2+(x_2-0.6)^2<0.02\}}(x)$$
- $d = 3, \quad u_1 = 0, \quad u_2 = 0.1, \quad u_3 \in \{0.2, 0.12\}$
- $y^\delta = y^\dagger + \xi, \quad \xi \in \mathcal{N}\left(y^\dagger, \delta\|y^\dagger\|_\infty\right)$
- finite element discretization: uniform grid, $256 \times 256$ nodes
- $\alpha = 5 \cdot 10^{-5}$, $\beta = 10^{-1}$ (no free arc)
- terminate at $\gamma < 10^{-12}$

# Numerical example: $u_3 = 0.2$



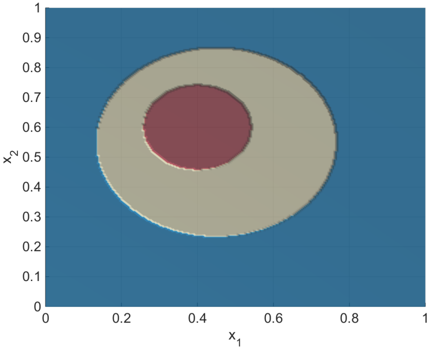




Figure: exact parameter $u^\dagger$

# Numerical example: $u_3 = 0.2$

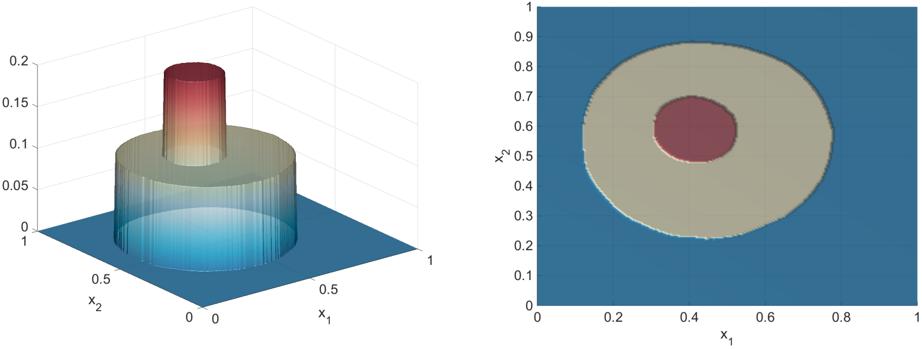



Figure: reconstruction $u^\delta$, $\delta = 0.1$

# Numerical example: $u_3 = 0.2$

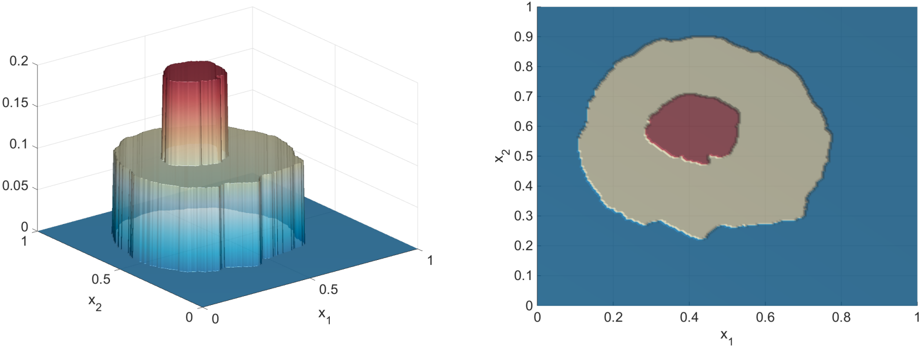



Figure: reconstruction $u^\delta$, $\quad \delta = 0.5$

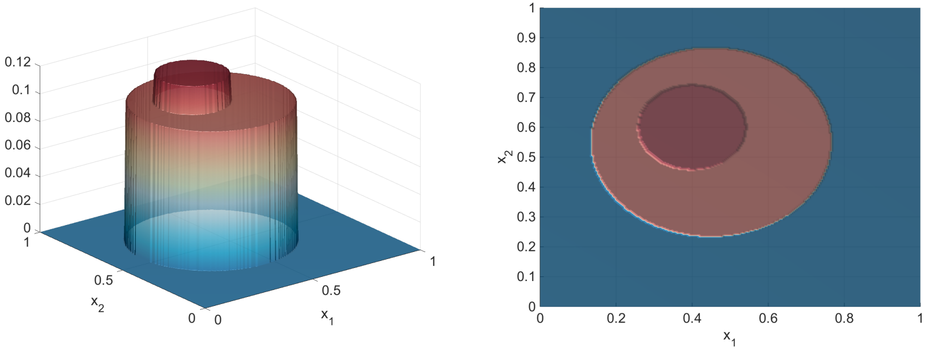
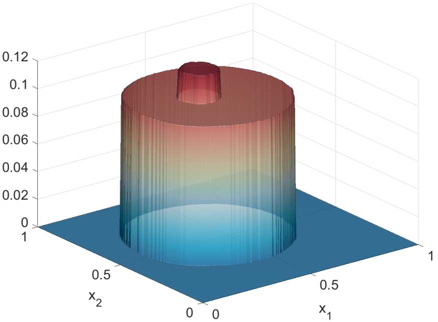



Figure: exact parameter $u^\dagger$

# Numerical example: $u_3 = 0.12$



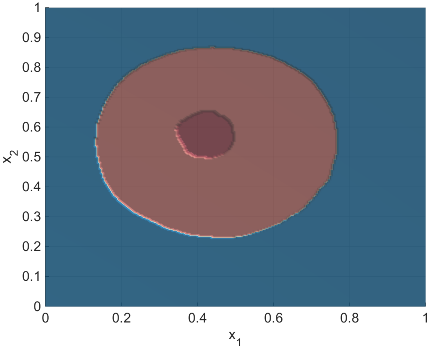




Figure: reconstruction $u^{\delta}$, $\delta = 0.1$

# Numerical example: $u_3 = 0.12$

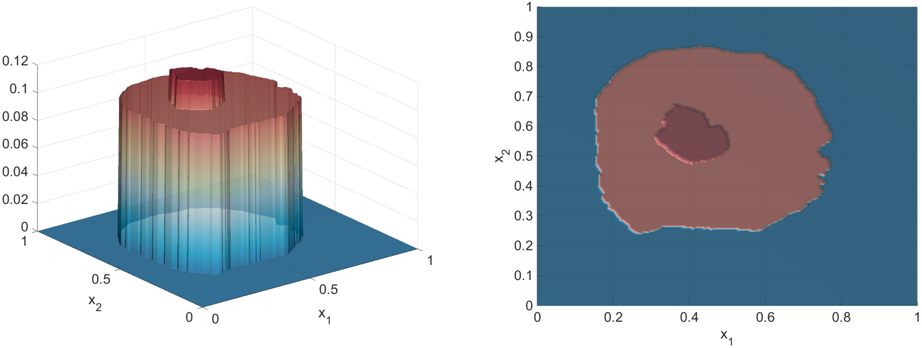



Figure: reconstruction $u^\delta$, $\delta = 0.5$

# Example: nonlinear inverse problem

- $S : u \mapsto y$ solving

$$-\Delta y + uy = f$$

- $\mathcal{F}$ nonconvex, but (modified) approach applicable

- numerical example: $\Omega = [0,1]^2, \quad f \equiv 1$

- $$u^\dagger(x) = u_1 + u_2\chi_{\{x:(x_1-0.45)^2+(x_2-0.55)^2<0.1\}}(x) + (u_3 - u_2)\chi_{\{x:(x_1-0.4)^2+(x_2-0.6)^2<0.02\}}(x)$$

- $y^\delta = S(u^\dagger) + \xi$

- $\alpha = 3 \cdot 10^{-5}, \quad \beta = \infty$ (formal), $\quad \gamma \to 10^{-12}$

# Numerical example: nonlinear problem

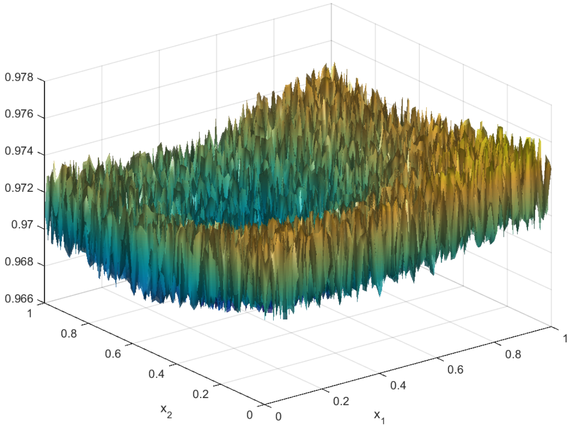
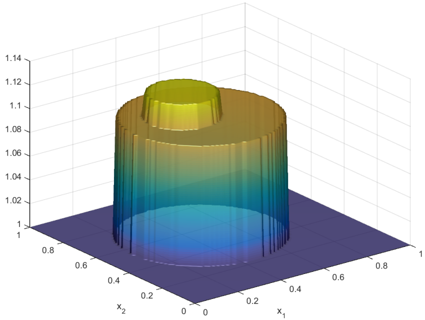



Figure: noisy data $y^\delta$

# Numerical example: nonlinear problem



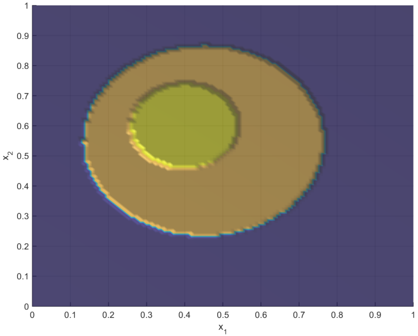




Figure: exact parameter $u^{\dagger}$

# Numerical example: nonlinear problem

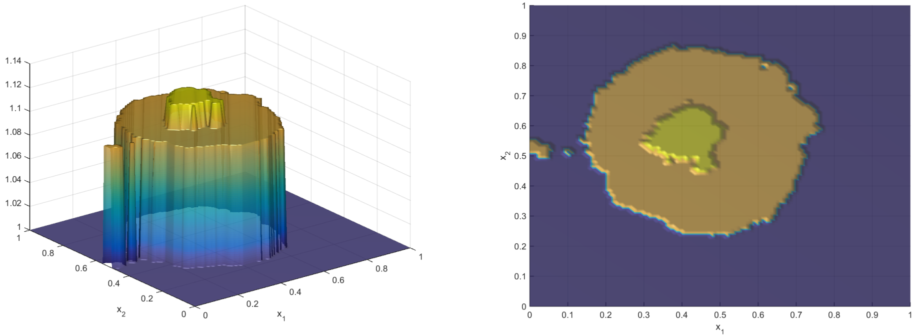



Figure: reconstruction $u^\delta$

# Conclusion

Convex relaxation of discrete regularization:

- well-posed primal-dual optimality system
- solution optimal under general assumptions
- linear complexity in number of parameter values
- $\rightsquigarrow$ efficient numerical solution (superlinear convergence)

Outlook:

- regularization properties, parameter choice
- nonlinear inverse problems: EIT
- combination with BV regularization
- other hybrid discrete–continuous problems

Preprint, MATLAB/Python codes:

`http://www.uni-due.de/mathematik/agclason/clason_pub.php`